

On Centres of 3-Blocks of the Ree groups ${}^2G_2(q)$

Julian Brough¹ and Inga Schwabrow²

¹ *FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany*
E-mail: brough@mathematik.uni-kl.de

² *School of Mathematics, The University of Manchester, Manchester, M13 9PL, UK*
E-mail: inga.schwabrow@gmx.de

Abstract

Let $G := {}^2G_2(q)$ be the simple Ree group with $q = 3^{2k+1}$ and k a positive integer. We show that the centre of the principal block $Z(kGe_0)$, where k is an algebraically closed field of characteristic 3, is not isomorphic to the centre of the Brauer corresponding block $Z(kN_G(P))$, where $N_G(P)$ is the normaliser in G of a Sylow 3-subgroup. As part of the proof, we compute the conjugacy classes of elements and the character tables of the maximal parabolic subgroups of G .

1 Introduction

Broué's conjecture postulates that if a block has abelian defect groups, then there exists a perfect isometry between the block and its Brauer correspondent. If such a perfect isometry exists, it follows that the centres of the blocks are isomorphic over a sufficiently large complete discrete evaluation ring \mathcal{O} . In particular, $Z(\mathcal{O}G\hat{e}_0) \cong Z(\mathcal{O}N_G(P)\hat{f}_0)$, for \hat{e}_0 and \hat{f}_0 the corresponding principle block idempotents.

Originally, Broué also made this conjecture for the principal blocks, in the case that the Sylow normaliser controls the p -fusion [2]. An initial counterexample to this conjecture was given by Broué and Serre [3, Section 6]. In particular, they considered the Cartan matrices to conclude that there is no derived equivalence in the case of ${}^2B_2(8)$ with $p = 2$.

Later on, Cliff elaborated on this counterexample by considering the radical structure of the centres of the blocks [6]. Let G be a finite simple group of Lie type ${}^2B_2(q)$ and P a Sylow 2-subgroup G , where $q = 2^{2k+1} \geq 8$. Cliff observed that over a field k of characteristic 2, $Z(kGe_0) \cong Z(kN_G(P))$, while over a discrete valuation ring \mathcal{O} of characteristic zero $Z(\mathcal{O}G\hat{e}_0) \not\cong Z(\mathcal{O}N_G(P))$ [6, Theorem 4.1]. This provided an infinite family of pairs of blocks whose centres are isomorphic over a field k of characteristic p , but are not perfectly isometric over a discrete valuation ring of characteristic 0.

In this paper we extend the ideas of Cliff to the simple groups of Lie type ${}^2G_2(q)$, where $q = 3^{2k+1}$, and show that in this case there does not exist an isomorphism of centres over a field k of characteristic 3, which implies there does not exist one over \mathcal{O} . To do this we analyse the radical structure of the centres of the principle blocks of ${}^2G_2(q)$ and $N_G(P)$ for P a Sylow 3-subgroup. Note that the group algebra $kN_G(P)$ is indecomposable. Although $Z(kGe_0)$ and $Z(kN_G(P))$ have the same dimension over k , we will show in Theorem 3.1 that $LL(Z(kGe_0)) = 3$, whereas Theorem 4.1 states that $LL(Z(kN_G(P))) = 2$; hence no such isomorphism can exist.

For a Sylow 3-subgroup P of ${}^2G_2(q)$, the normaliser is isomorphic to $P \rtimes C_{q-1}$ [17]. Therefore $N_G(P)$ is a solvable group; furthermore $O_{3'}(N_G(P)) = 1$ and so $N_G(P)$ has a unique 3-block b [13, Proposition III.1.12]. This implies that the principle block of ${}^2G_2(q)$, B_0 , is the unique 3-block of maximal defect. Moreover as ${}^2G_2(q)$ has trivial intersection Sylow 3-subgroups [9, p. 307], its blocks either have maximal defect or defect zero. However ${}^2G_2(q)$ only has one character of 3-defect zero (the Steinberg character), thus the group ${}^2G_2(q)$ has two blocks. The group ${}^2G_2(q)$ has $q + 8$ irreducible characters [17, p. 85], from which it now follows by [1, Proposition 6.2] that $k(B_0) = q + 7 = k(b)$, where $k(B)$ is the number of irreducible complex characters in a block B .

In order to study the Loewy length of the center of these group algebras, we make use of the character tables. The (complex) irreducible characters of ${}^2G_2(q)$ were computed by Ward [17]. For the normaliser of a Sylow 3-subgroup, we use the character degrees obtained by Eaton [8] and a

partial segment of the character table computed by Gramain [10] to produce the full character table, Table 2. Note that a complete character table for $N_G(P)$ was given by Landrock and Michler [14, p. 88]. However as only vague details were provided, we give an independent and much more detailed construction of the character table here.

We summarise the main result obtained in this paper.

Theorem 1.1. *Let k be an algebraically closed field of characteristic 3 and $G = {}^2G_2(q)$, $q = 3^{2k+1} \geq 27$, $P \in \text{Syl}_3(G)$. Then $LL(Z(kGe_0)) = 3 > 2 = LL(Z(kN_G(P)))$; hence*

$$Z(kGe_0) \not\cong Z(kN_G(P)).$$

As an immediate corollary, we get the following result.

Corollary 1.2. *There is no perfect isometry, and hence no derived equivalence, between kGe_0 and $kN_G(P)$.*

2 Preliminary

Let (K, \mathcal{O}, k) be a p -modular system; that is, K is a field of characteristic zero, \mathcal{O} is a complete valuation ring with unique maximal ideal $J(\mathcal{O})$, and k is a field of characteristic p . In addition $\mathcal{O}/J(\mathcal{O}) \cong k$ and K is the field of fractions of \mathcal{O} .

2.1 The class algebra constants

The conjugacy class sums form a basis of $Z(kG)$, and therefore the product of any conjugacy class sums must be a sum of conjugacy class sums. Let $\mathcal{C}(x)$ denote the conjugacy class of x and

$$\widehat{\mathcal{C}}(x) := \sum_{g \in \mathcal{C}(x)} g$$

the class sum of $\mathcal{C}(x)$ which is an element in \mathcal{OG} . Then the following common notation is adopted:

$$\widehat{\mathcal{C}}(x)\widehat{\mathcal{C}}(y) = \sum_{z \in \mathcal{P}} a(x, y, z)\widehat{\mathcal{C}}(z) \quad \text{for } x, y \in G, \quad (1)$$

where \mathcal{P} is a set of representatives of the conjugacy classes of G and the constants $a(x, y, z)$ are referred to as the class algebra constants. We note that from the definition, it follows that the structure constants $a(x, y, z)$ lie in \mathbb{Z} .

Burnside's original work in representation theory over \mathbb{C} provided a method for obtaining the class algebra constants from the character table of the group. In particular, this connection is made precise by Burnside's formula [4, p.316] which forms a crucial part in the study of representation theory and will play a large role in the calculations to follow in this paper.

Given $x, y, z \in G$, Burnside's formula states:

$$a(x, y, z) = \frac{|G|}{|C_G(x)||C_G(y)|} \sum_{\theta \in \text{Irr}(G)} \frac{\theta(x)\theta(y)\theta(z^{-1})}{\theta(1)}. \quad (2)$$

From this formula it is clear that $a(x, y, z) = a(y, x, z)$. Additionally we make the following observation, which reduces the number of explicit calculations required to compute all the structure constants.

Lemma 2.1. *Given $x, y, z \in G$, then $a(x, y, z) = a(x^{-1}, y^{-1}, z^{-1})$. Furthermore*

$$a(x, y, z) = \frac{|C_G(z)|}{|C_G(y)|} a(z^{-1}, y, x^{-1}) = \frac{|C_G(z)|}{|C_G(y)|} a(x^{-1}, z, y).$$

Proof. As $xy = z$ if and only if $y^{-1}x^{-1} = z^{-1}$, it follows that $a(x, y, z) = a(y^{-1}, x^{-1}, z^{-1}) = a(x^{-1}, y^{-1}, z^{-1})$. We note that the second statement follows by swapping either $\theta(x)$ with $\theta(z)$ or $\theta(y)$ with $\theta(z)$ and taking the complex conjugate in Burnside's formula. \square

As we are working over a field of positive characteristic, we are interested when the field characteristic divides the structure constant $a(x, y, z)$. Using the notion of the defect of a conjugacy class provides one method to determine this in certain cases.

Definition 2.2. Let g be an element of a finite group G . Then the p -defect of the conjugacy class of g in G is given by d_g , where p^{d_g} is the order of a Sylow p -subgroup of $C_G(g)$.

The following corollary follows from Lemma 2.1.

Corollary 2.3. [7, Cor 87.7] Let k be a field of characteristic p , and for $g \in G$, let d_g be the defect of the conjugacy class $\mathcal{C}(g)$. If $d_y < d_z$ or $d_x < d_z$ then $p \mid a(x, y, z)$. Therefore

$$\widehat{\mathcal{C}}(x)\widehat{\mathcal{C}}(y) = \sum_{d_z \leq \min\{d_x, d_y\}} a(x, y, z)\widehat{\mathcal{C}}(z) \in kG.$$

Proposition 2.4. Fix $y, z \in G$. Then

$$\sum_{x \in \mathcal{P}} a(x, y, z) = |\mathcal{C}(y)|.$$

Proof. We have

$$\sum_{x \in \mathcal{P}} \widehat{\mathcal{C}}(x)\widehat{\mathcal{C}}(y) = \sum_{g \in G} g\widehat{\mathcal{C}}(y) = \sum_{y' \in \mathcal{C}(y)} \widehat{G}y' = |\mathcal{C}(y)| \sum_{g \in G} g = |\mathcal{C}(y)| \sum_{z \in \mathcal{P}} \widehat{\mathcal{C}}(z).$$

On the other hand

$$\sum_{x \in \mathcal{P}} \widehat{\mathcal{C}}(x)\widehat{\mathcal{C}}(y) = \sum_{x \in \mathcal{P}} \sum_{z \in \mathcal{P}} a(x, y, z)\widehat{\mathcal{C}}(z).$$

Therefore the coefficient of $\widehat{\mathcal{C}}(z)$ is $\sum_{x \in \mathcal{P}} a(x, y, z)$ and the proposition follows. \square

We end this section with a result which relates the structure constants of a group G , with trivial intersection Sylow p -subgroups, to the normaliser of Q a Sylow p -subgroup. In particular, we generalise the result of Cliff for the Suzuki groups [6, Lemma 3.2] by observing that the argument holds whenever the Sylow p -subgroups have the trivial intersection property. Note that for such a group G , we have that $N_G(Q)$ controls fusion of Q in G ; that is, if $x^g \in Q$ for $x \in Q$ and $g \in G$, then either $x = 1$ or $g \in N_G(Q)$. This follows from the observation that $x^g \in Q \cap Q^g$ which equals Q or is trivial.

Proposition 2.5. Let G be a finite group with trivial intersection Sylow p -subgroups. Let $x, y, z \in Q \setminus \{1_G\}$ where $Q \in \text{Syl}_p(G)$. Then $a(x, y, z) \equiv a_H(x, y, z) \pmod{|C_G(z)|_p}$ where $H = N_G(Q)$.

Proof. Let

$$\mathcal{A} = \{(x', y') \mid x' \in x^G, y' \in y^G, x'y' = z\}.$$

Then $|\mathcal{A}| = a(x, y, z)$ and \mathcal{A} can be split into two disjoint sets $\mathcal{A}_1 \cup \mathcal{A}_2$, where

$$\begin{aligned} \mathcal{A}_1 &= \{(x', y') \mid x' \in x^G \cap Q, y' \in y^G, x'y' = z\} \\ \mathcal{A}_2 &= \{(x', y') \mid x' \in x^G \setminus Q, y' \in y^G, x'y' = z\} \end{aligned}$$

Note that as $z = x'y' \in Q$, we have $x' \in x^G \cap Q$ if and only if $y' \in y^G \cap Q$.

Claim 1 $x^G \cap Q = x^{N_G(Q)}$

Suppose $x' \in x^G \cap Q$. Then there exists an element $g \in G$ such that $x^g = x' \in Q \cap x^G$. Since the Sylow p -subgroups of G are trivial intersection, by the above remark, there exists $h \in N_G(Q)$ such that $x^g = x^h$. Hence $x' = x^g = x^h \in x^{N_G(Q)}$.

Therefore $|\mathcal{A}_1| = a_H(x, y, z)$.

Claim 2 The size of \mathcal{A}_2 is divisible by $|C_G(z)|_p$.

Take an element z in $Q \setminus \{1\}$. Suppose $g \in N_G(\langle z \rangle)$, so $\langle z \rangle^g = \langle z \rangle$; this implies $\langle z \rangle^g \in Q \cap Q^g = \{1\}$, a contradiction unless $g \in N_G(Q)$. Hence the normaliser and thus the centraliser of z is a subgroup of $N_G(Q)$, i.e. $C_G(z) \leq N_G(\langle z \rangle) \leq N_G(Q)$.

Note that if $g \in C_G(z)$ then $z^g = z = (x'y')^g = x'^g y'^g$; since $C_G(z) \leq N_G(Q)$, if $(x')^g \in Q$ then $x' \in Q^g = Q$. Therefore $C_G(z)$ acts on \mathcal{A}_2 .

Suppose $(x', y') \in \mathcal{A}_2$. By the orbit stabiliser theorem, the size of a $C_G(z)$ -orbit containing (x', y') is given by

$$\begin{aligned} |(x', y')^{C_G(z)}| &= [C_G(z) : (C_{C_G(z)}(x') \cap C_{C_G(z)}(y'))] \\ &= [C_G(z) : (C_G(z) \cap C_G(x') \cap C_G(z) \cap C_G(y'))] \end{aligned}$$

Consider $C_G(z) \cap C_G(x')$, and suppose $x' = x^g$ such that $g \notin N_G(Q)$; then $C_G(z) \cap C_G(x^g) \leq N_G(Q) \cap N_G(Q^g)$. Suppose $S \in \text{Syl}_p(N_G(Q) \cap N_G(Q^g))$. As $S \leq N_G(Q)$ it follows that $S \leq Q$; similarly $S \leq N_G(Q^g)$ implies $S \leq Q^g$. Hence as $g \notin N_G(Q)$, $S \leq Q \cap Q^g = 1 \setminus \{1\}$, and it follows that p does not divide $|C_G(z) \cap C_G(x')|$. In particular, p does not divide $|C_G(z) \cap C_G(x') \cap C_G(y')|$.

Hence $|(x', y')^{C_G(z)}|_p = |C_G(z)|_p$ for all pairs $(x', y') \in \mathcal{A}_2$ and therefore $|C_G(z)|_p$ divides the size of \mathcal{A}_2 .

Finally, combining the two claims, $|\mathcal{A}| \equiv |\mathcal{A}_1| \pmod{|C_G(z)|_p}$, and since \mathcal{A}_1 and \mathcal{A}_2 are disjoint, $|\mathcal{A}| \equiv |\mathcal{A}_1| \equiv a_H(x, y, z) \pmod{|C_G(z)|_p}$. \square

2.2 The character table of the Sylow normaliser

In this section we shall construct the full complex character table for the normaliser of a Sylow 3-subgroup P of ${}^2G_2(q)$ with $q = 3^{2k+1}$. First we recall the notation used to describe both the Sylow 3-subgroup and its normaliser. The standard references for the description presented are [5] and [10].

Let P be a Sylow 3-subgroup of G . Then P is isomorphic to the set of elements $\{x(t, u, v) \mid t, u, v \in \mathbb{F}_q\}$ endowed with a multiplication given by

$$x(t_1, u_1, v_1)x(t_2, u_2, v_2) = x(t_1 + t_2, u_1 + u_2 - t_1 t_2^{3\theta}, v_1 + v_2 - t_2 u_1 + t_1 t_2^{3\theta+1} - t_1^2 t_2^{3\theta})$$

where θ is the automorphism of \mathbb{F}_q given by $\lambda^\theta = \lambda^{3^k}$ for all $\lambda \in \mathbb{F}_q$. Observe that the inverse of an element $x(t, u, v) \in P$ is given by

$$x(t, u, v)^{-1} = x(-t, -u - t^{3\theta+1}, -v - tu + t^{3\theta+2}).$$

As P is a normal Sylow 3-subgroup of $N_G(P)$, there exists a complement W to P in $N_G(P)$. In particular, W is a cyclic group of order $q - 1$ that can be labelled by the set $\{h(w) \mid w \in \mathbb{F}_q^\times\}$; furthermore the conjugation of P by W is given by

$$h(w)x(t, u, v)h(w)^{-1} = x(w^{2-3\theta}t, w^{3\theta-1}u, wv)$$

where $t, u, v, w \in \mathbb{F}_q, w \neq 0$. Note that $h(1) = 1_W = 1_G$ and $h(-1)$ is the unique involution in W .

The elements of P form 7 conjugacy classes in $N_G(P)$ and the class sizes and element orders can be found in [10, Section 2.5.2]. Since there are $q + 7$ conjugacy classes overall in $N_G(P)$, it remains to determine the remaining q conjugacy classes. The following details are summarised in Table 1.

As W is cyclic, to construct $C_N(h(w))$, it is enough to find $x \in C_P(h(w))$; moreover $\mathcal{C}(h(w)) = A_w h(w)$ for some subset $A_w \subset P$. It is clear that $x(t, u, v) = h(w)x(t, u, v)h(w)^{-1} = x(w^{2-3\theta}t, w^{3\theta-1}u, wv)$ implies $v = 0$ and $w^{2-3\theta} = w^{3\theta-1} = 1$. However $w^{2-3\theta} = 1$ has a unique solution $w = 1$ in \mathbb{F}_q ; while $w^{3\theta-1} = 1$ only has two solutions $w = \pm 1$ [10, p. 62]. Thus if $w \neq \pm 1$, then $C_N(h(w)) = W$ and therefore $|A_w| = |P|$ and $\mathcal{C}(h(-1)) = Ph(-1)$. If $w = -1$, then $C_P(h(-1)) = \{x(0, u, 0) \mid u \in \mathbb{F}_q\}$ and so $|C_N(h(-1))| = q(q - 1)$. Additionally, as $x(t, u, v)h(-1)x(t, u, v)^{-1} = x(t, t^{3\theta+1}, v + tu)h(-1)$, it follows that the q^2 elements $x(t, t^{3\theta+1}, v + tu)h(-1)$ form the conjugacy class $\mathcal{C}(h(-1))$. We note that $x(0, -1, 0)h(-1) \notin \mathcal{C}(h(-1))$.

Next consider $x(0, -1, 0)h(-1) \in N_G(P)$. If $x(t, u, v) \in C_P(x(0, -1, 0)h(-1))$ then $x(t, u, v) = x(-t, -1 + u + w^{3\theta-1}, -v - t)$. In particular, $t = 0$ and so $v = 0$, which further implies $w^{3\theta-1} = 1$.

Hence $C_N(x(0, -1, 0)h(-1)) = \{x(0, u, 0)h(w) \in N_G(P) \mid u \in \mathbb{F}_q, w = \pm 1\}$ which has size $2q$. Note that $(x(0, -1, 0)h(-1))^{-1} = h(-1)x(0, 1, 0) = x(0, 1, 0)h(-1)$. However as

$$x(t, u, v)h(w)x(0, -1, 0)h(-1)h(w)^{-1}x(t, u, v)^{-1} = x(-t, -w^{3\theta-1} + t^{3\theta+1}, -vt^{3\theta+2} + tw^{3\theta-1})h(-1),$$

it follows that $(x(0, -1, 0)h(-1))^{-1} = x(0, 1, 0)h(-1) \notin \mathcal{C}(x(0, -1, 0)h(-1))$, otherwise $t = 0$ and $-w^{3\theta-1} = 1$, which has no solution $w \in \mathbb{F}_q^\times$ as 4 does not divide $q - 1$. Hence the two remaining conjugacy classes are an inverse pair.

Table 1: Conjugacy classes of $N_G(P)$

label	$\mathcal{C}(g)$	$o(g)$	$ C_N(g) $	$C_N(g)$	$ \mathcal{C}(g) $
1_N	$x(0, 0, 0)h(1)$	1	$ N $	$N_G(P)$	1
$X = Z(P) \setminus \{1_P\}$	$x(0, 0, v)h(1)$ $v \neq 0$	3	q^3	P	$q - 1$
T, T^{-1}	$x(0, u, v)h(1)$ $u \neq 0$	3	$2q^2$	$x(0, u, v)h(w)$ $w = \pm 1$	$q(q - 1)/2$
Y, YT, YT^{-1}	$x(t, u, v)h(1)$ $t \neq 0$	9	$3q$	$x(0, 0, v_2),$ $x(t, u, v_2),$ $x(-t, -t^{3\theta+1} - u, v_2)$ where $v_2 \in \mathbb{F}_q$	$q^2(q - 1)/3$
$Ph(w)$ $w \neq \pm 1$	$x(t, u, v)h(w)$ $w \text{ fixed, } w \neq \pm 1$	$ h(w) $	$q - 1$	$x(0, 0, 0)h(w)$	q^3
$J : x(0, 0, 0)h(-1)$	$x(t, t^{3\theta+1}, v + tu)h(-1)$	2	$q(q - 1)$	$x(0, u, 0)h(w)$	q^2
JT	α	6	$2q$	$x(0, u, 0)h(w)$ $w = \pm 1$	$q^2(q - 1)/2$
$x(0, -1, 0)h(-1)$					
JT^{-1}	β	6	$2q$	$x(0, u, 0)h(w)$ $w = \pm 1$	$q^2(q - 1)/2$
$(x(0, -1, 0)h(-1))^{-1}$					

where $t, u, v \in \mathbb{F}_q$ and $w \in \mathbb{F}_q^\times$,

$$\alpha = x(-t, -w^{3\theta-1} + t^{3\theta+1}, -v - t^{3\theta+1} + tw^{3\theta-1})h(-1)$$

and

$$\beta = x(-t, w^{3\theta-1} + t^{3\theta+1}, -v - t^{3\theta+1} - tw^{3\theta-1})h(-1).$$

2.2.1 Detailed construction of the character table

In this section we make use of the common notation $\bar{\theta}$ to denote the complex conjugate of the character θ . Furthermore given two characters θ_1 and θ_2 of $N_G(P)$, then $\langle \theta_1, \theta_2 \rangle$ denotes the inner product of the two characters of $N_G(P)$:

$$\langle \theta_1, \theta_2 \rangle := \frac{1}{|N_G(P)|} \sum_{g \in N_G(P)} \theta_1(g) \overline{\theta_2(g)}.$$

Gramain [10, Section 2.5.4] gives the following characters of $N_G(P)$ which are induced from characters in P .

$\text{Irr}(P)$	$\text{Ind}_P^{N_G(P)}$	$\text{Irr}(N_G(P))$
$\{\lambda_2, \dots, \lambda_q\}$	\rightarrow	λ , degree $q - 1$
$\{\psi_2, \dots, \psi_q\}$	\rightarrow	ψ , degree $(q - 1)q$
$\{\chi_{i,j}, i = 1, 2 \text{ and } 1 \leq j \leq \frac{q-1}{2}\}$	\rightarrow	χ , degree $(q - 1)3^k$
$\{\overline{\chi}_{i,j}, i = 1, 2 \text{ and } 1 \leq j \leq \frac{q-1}{2}\}$	\rightarrow	$\overline{\chi}$, degree $(q - 1)3^k$
$\{\chi_{3,j}, 1 \leq j \leq \frac{q-1}{2}\}$	\rightarrow	$\mu_1 + \mu_2$, each μ_i irreducible of degree $\frac{q-1}{2}3^k$
$\{\overline{\chi}_{3,j}, 1 \leq j \leq \frac{q-1}{2}\}$	\rightarrow	$\overline{\mu_1} + \overline{\mu_2}$, each $\overline{\mu}$ irreducible of degree $\frac{q-1}{2}3^k$

In addition to these irreducible characters we obtain $q - 1$ linear characters $\alpha_0 = 1_N, \dots, \alpha_{q-2}$ for $N_G(P)$ by lifting the characters $\tilde{\alpha}_i$ with $0 \leq i \leq q - 2$ from the quotient group $N_G(P)/P \cong C_{q-1}$. In other words $\alpha_i(g) := \tilde{\alpha}_i(gP)$, so $\alpha_i|_P = 1$. Let $W = \langle h \rangle$, and $\tilde{\alpha}_i$ defined by $\tilde{\alpha}_i(h^j) = \xi^{ij}$ for ξ a fixed primitive $(q-1)^{th}$ root of unity. As the coset $J \cdot P$ is the unique involution in W and $J \cdot P = JT^{\pm 1} \cdot P$, it follows that $\alpha_i(JT^{\pm 1}) = \alpha_i(J) = \tilde{\alpha}_i(h^{\frac{q-1}{2}}) = (-1)^i$. We now fix the elements w_j such that $Ph(w_j)$ is mapped to h^j in W . For such a labeling we have $\alpha_i(Ph(w_j)) = \xi^{ij}$. Also note that $w_{\frac{q-1}{2}} = -1$ and $Ph(-1)$ labels the union of the conjugacy classes of J, JT and JT^{-1} . From now on, when we write $Ph(w_j)$ we exclude the case that $w_j = \pm 1$ unless explicitly stated otherwise.

The following table combines the details contained in [8] (character degrees) and [10] (character values) with the above values for the linear characters α_i .

$ \mathcal{C}(g) $	1	$q - 1$	$\frac{q(q-1)}{2}$	$\frac{q^2(q-1)}{3}$	$\frac{q^2(q-1)}{3}$	$\frac{q^2(q-1)}{3}$	q^3	q^2	$\frac{q^2(q-1)}{2}$
$ C_N(g) $	$q^3(q-1)$	q^3	$2q^2$	$3q$	$3q$	$3q$	$q-1$	$q(q-1)$	$2q$
	1	X	T, T^{-1}	Y	YT	YT^{-1}	$Ph(w_j)$	J	JT, JT^{-1}
$\alpha_0 = 1_N$	1	1	1	1	1	1	1	1	1
α_i	1	1	1	1	1	1	ξ^{ij}	$(-1)^i$	$(-1)^i$
λ	$q - 1$			-1	-1	-1			
μ_1	$\frac{3^k(q-1)}{2}$			$-\varepsilon 3^k$	$-\varepsilon 3^k \bar{\omega}$	$-\varepsilon 3^k \omega$			
μ_2	$\frac{3^k(q-1)}{2}$			$-\varepsilon 3^k$	$-\varepsilon 3^k \bar{\omega}$	$-\varepsilon 3^k \omega$			
$\bar{\mu}_1$	$\frac{3^k(q-1)}{2}$			$-\varepsilon 3^k$	$-\varepsilon 3^k \omega$	$-\varepsilon 3^k \bar{\omega}$			
$\bar{\mu}_2$	$\frac{3^k(q-1)}{2}$			$-\varepsilon 3^k$	$-\varepsilon 3^k \omega$	$-\varepsilon 3^k \bar{\omega}$			
χ	$3^k(q-1)$			$\varepsilon 3^k$	$\varepsilon 3^k \bar{\omega}$	$\varepsilon 3^k \omega$			
$\bar{\chi}$	$3^k(q-1)$			$\varepsilon 3^k$	$\varepsilon 3^k \omega$	$\varepsilon 3^k \bar{\omega}$			
ψ	$q(q-1)$			0	0	0			

for some fixed $\varepsilon \in \{\pm 1\}$, and where $q = 3^{2k+1}$, $\omega = e^{2i\pi/3}$, and ξ is a fixed primitive $(q-1)^{th}$ root of unity.

In order to fill in the remaining entries, we apply the orthogonality relations of a character table. First we consider the characters ψ, λ and χ which arise as induced characters. Therefore $\psi(g) = \lambda(g) = \chi(g) = \bar{\chi}(g) = 0$ for all $g \in N_G(P) \setminus P$.

The characters ψ, λ and χ

As $\psi|_P = \sum_{i=2}^q \psi_i$ and $\psi_i(g) = 0$ for all $g \in P \setminus Z(P)$ [10, p.70], it follows that $\psi(g) = 0$ for all $g \in P \setminus Z(P)$. Moreover

$$\langle \psi, \alpha_0 \rangle = \frac{1}{|N_G(P)|} (q(q-1) + \psi(X)(q-1)) = 0$$

implies $\psi(X) = -q$.

Now that we have ψ , both $\chi(X)$ and $\lambda(X)$ can be computed:

$$\langle 0 = \theta, \psi \rangle = \begin{cases} \frac{1}{|N_G(P)|} (q(q-1)(q-1) - q\lambda(X)(q-1)) & \theta = \lambda; \\ \frac{1}{|N_G(P)|} (q(q-1)3^k(q-1) - q(q-1)\chi(X)) & \theta = \chi. \end{cases}$$

Thus $\lambda(X) = q - 1$ and $\chi(X) = \bar{\chi}(X) = 3^k(q-1)$. Therefore it remains to compute $\lambda(T)$ and $\chi(T)$. Since λ is the unique character taking the value -1 on Y , $\lambda = \bar{\lambda}$. In other words λ is real valued and so $\lambda(T^{-1}) = \bar{\lambda}(T) = \lambda(T)$. Hence,

$$\langle \lambda, \alpha_0 \rangle = \frac{1}{|N_G(P)|} \left(q - 1 + (q-1)^2 + 2\lambda(T) \frac{q(q-1)}{2} - 3 \frac{q^2(q-1)}{3} \right) = 0$$

and it follows that $\lambda(T) = \lambda(T^{-1}) = q - 1$.

To compute $\chi(T)$, we need to make use of two relations:

$$\langle \chi, \alpha_0 \rangle = \frac{1}{|N_G(P)|} \left(3^k(q-1)(1+q-1) + \frac{q(q-1)}{2} (\chi(T) + \bar{\chi}(T)) + \frac{q^2(q-1)}{3} \cdot \varepsilon \cdot 3^k(1+\omega+\bar{\omega}) \right) = 0$$

and

$$\langle \chi, \chi \rangle = \frac{1}{|N_G(P)|} \left((3^k(q-1))^2(1+q-1) + 2\chi(T)\overline{\chi(T)}\frac{q(q-1)}{2} + \frac{q^2(q-1)}{3} \cdot (\varepsilon \cdot 3^k)^2(1+\omega\overline{\omega} + \overline{\omega}\omega) \right) = 1.$$

Hence $\chi(T) + \overline{\chi(T)} = -2 \cdot 3^k$ and $\chi(T)\overline{\chi(T)} = q(q-2 \cdot 3^{2k}) + 3^{2k} = 3^{2k}(q+1)$, where the finally equality comes from substituting in $q = 3^{2k+1}$. Thus $\chi(T)$ and $\overline{\chi(T)} = \chi(T^{-1})$ are the zeros of the polynomial $x^2 + (\chi(T) + \overline{\chi(T)})x + \chi(T)\overline{\chi(T)}$. In particular, $\chi(T) = -3^k + 3^{2k}\sqrt{-3}$ and $\chi(T^{-1}) = -3^k - 3^{2k}\sqrt{-3}$.

So far we have calculated the following additional entries:

$ \mathcal{C}(g) $	$q-1$	$\frac{q(q-1)}{2}$	$\frac{q(q-1)}{2}$	q^3	q^2	$\frac{q^2(q-1)}{2}$	$\frac{q^2(q-1)}{2}$
$ C_N(g) $	q^3	$2q^2$	$2q^2$	$q-1$	$q(q-1)$	$2q$	$2q$
	X	T	T^{-1}	$Ph(w)$	J	JT	JT^{-1}
λ	$q-1$	$q-1$	$q-1$	0	0	0	0
χ	$3^k(q-1)$	$-3^k + 3^{2k}\sqrt{-3}$	$-3^k - 3^{2k}\sqrt{-3}$	0	0	0	0
$\overline{\chi}$	$3^k(q-1)$	$-3^k - 3^{2k}\sqrt{-3}$	$-3^k + 3^{2k}\sqrt{-3}$	0	0	0	0
ψ	$-q$	0	0	0	0	0	0

where $q = 3^{2k+1}$.

The characters μ_i and $\overline{\mu_i}$ for $i \in \{1, 2\}$

As before, to determine $\mu_i(X)$, we take the inner product of μ_i and ψ :

$$\langle \mu_i, \psi \rangle = \frac{1}{|N_G(P)|} \left(q(q-1)\frac{3^k(q-1)}{2} - q(q-1)\mu_i(X) \right) = 0$$

and thus $\mu_i(X) = \frac{3^k(q-1)}{2}$.

Consider $\mu_i(Ph(w_j))$, for $w_j \neq \pm 1$. Since 4 does not divide $q-1$, $\alpha_k(Ph(w_j))$ takes at least 5 distinct values for $0 \leq k \leq q-2$. Therefore if $\mu_i(Ph(w_j)) \neq 0$ then the set $\{\alpha_k\mu_i \mid 0 \leq k \leq q-2\}$ must contain 5 distinct irreducible characters of degree equal to that of μ_i , which is a contradiction. Hence $\mu_1(Ph(w_j)) = \mu_2(Ph(w_j)) = \overline{\mu_1}(Ph(w_j)) = \overline{\mu_2}(Ph(w_j)) = 0$.

As $\mathcal{C}(J) = \mathcal{C}(J^{-1})$, it follows that $\mu_i(J) \in \mathbb{R}$. Moreover JT and JT^{-1} are inverses and hence $\mu_i(JT^{-1}) = \overline{\mu_i}(JT)$. Furthermore $\mu_1 + \mu_2 = \text{Ind}_P^{N_G(P)}\theta$ which evaluates to zero on J, JT and JT^{-1} , and so $\mu_2(g) = -\mu_1(g)$, for $g = J, JT$ or JT^{-1} . We have the following part of the character table:

	J	JT	JT^{-1}
α_i	$(-1)^i$	$(-1)^i$	$(-1)^i$
λ	0	0	0
μ_1	c	b	\overline{b}
μ_2	$-c$	$-b$	$-\overline{b}$
$\overline{\mu_1}$	c	\overline{b}	b
$\overline{\mu_2}$	$-c$	$-\overline{b}$	$-b$
$\chi, \overline{\chi}, \psi$	0	0	0

where $c \in \mathbb{R}$ and $b \in \mathbb{C} \setminus \mathbb{R}$.

By column orthogonality

$$\sum_{\theta \in \text{Irr}(N_G(P))} \theta(J)^2 = q-1 + 4c^2 = q(q-1),$$

and thus $c = (q-1)/2$.

By using column orthogonality again

$$\sum_{\theta \in \text{Irr}(N_G(P))} \theta(JT)\overline{\theta(JT)} = q-1 + 4(b\overline{b}) = 2q$$

and so $b\bar{b} = \frac{q+1}{4}$. However using column orthogonality for JT and J yields

$$\sum_{\theta \in \text{Irr}(N_G(P))} \theta(JT)\theta(J) = q - 1 + b(q - 1) + \bar{b}(q - 1) = 0$$

and so $b + \bar{b} = -1$. Thus as with χ above, b and \bar{b} are the zeros of the polynomial $x^2 + (b + \bar{b})x + m\bar{b}$ and so it follows that $b = \frac{-1-3^k\sqrt{-3}}{2}$, by relabeling T and T^{-1} if necessary.

It only remains to compute μ_i on T and T^{-1} . Similar to the calculations for χ ,

$$\langle \mu_1, \theta \rangle \Rightarrow \begin{cases} \mu_1(T) + \overline{\mu_1(T)} = -3^k & \text{when } \theta = \alpha_0 \\ \mu_1(T)\overline{\mu_1(T)} = \frac{1}{4}(3^{4k+1} + 3^{2k}) & \text{when } \theta = \mu_1 \end{cases}$$

and solving the quadratic polynomial as before yields $\mu_1(T) = \frac{3^k \pm 3^{2k}\sqrt{-3}}{2}$.

Taking column orthogonality for T and J , we see that $\mu_1(T) + \bar{\mu}_1(T) = \mu_2(T) + \bar{\mu}_2(T)$. Row orthogonality implies $\langle \mu_1, \mu_1 \rangle = \langle \mu_2, \mu_2 \rangle = 1$. However as $\mu_2 = -\mu_1$ on J, JT and JT^{-1} , it follows that $\mu_1(T)\bar{\mu}_1(T) = \mu_2(T)\bar{\mu}_2(T)$. Hence $\mu_2(T) = \mu_1(T)$ or $\bar{\mu}_1(T)$. Finally, by column orthogonality for T and JT , we see that $(\mu_1(T) - \mu_2(T))\bar{b} = (\bar{\mu}_2(T) - \bar{\mu}_1(T))b$ and therefore it follows that if $\mu_2(T) = \bar{\mu}_1(T)$ then $\bar{b} = -b$, which is a contradiction. Thus $\mu_1(T) = \mu_2(T)$ and $\mu_1(T) = \mu_2(T) = \frac{-3^k + 3^{2k}\sqrt{-3}}{2}$.

Combining all the character values together, we have proven the following theorem.

Theorem 2.6. *Let $G = {}^2G_2(q)$ where $q = 3^{2k+1}$, and let $N_G(P)$ be the normaliser of a Sylow 3-subgroup. Then the character table of $N_G(P)$ is given by Table 2.*

Table 2: Character table of $N_G(P)$

$ C_N(g) $	$q^3(q-1)$	q^3	$2q^2$	$2q^2$	$3q$	$3q$	$3q$	$q-1$	$q(q-1)$	$2q$	$2q$
	1	X	T	T^{-1}	Y	YT	YT^{-1}	$Ph(w_j)$	J	JT	JT^{-1}
$\alpha_0 = 1_N$	1	1	1	1	1	1	1	1	1	1	1
α_i for $1 \leq i \leq q-2$	1	1	1	1	1	1	1	ξ^{ij}	$(-1)^i$	$(-1)^i$	$(-1)^i$
λ	$q-1$	$q-1$	$q-1$	$q-1$	-1	-1	-1	0	0	0	0
μ_1	$\frac{3^k(q-1)}{2}$	$\frac{3^k(q-1)}{2}$	$\frac{-3^k+3^{2k}\sqrt{-3}}{2}$	$\frac{-3^k-3^{2k}\sqrt{-3}}{2}$	$-\varepsilon 3^k$	$-\varepsilon 3^k \bar{\omega}$	$-\varepsilon 3^k \omega$	0	$\frac{q-1}{2}$	$\frac{-1-3^k\sqrt{-3}}{2}$	$\frac{-1+3^k\sqrt{-3}}{2}$
μ_2	$\frac{3^k(q-1)}{2}$	$\frac{3^k(q-1)}{2}$	$\frac{-3^k+3^{2k}\sqrt{-3}}{2}$	$\frac{-3^k-3^{2k}\sqrt{-3}}{2}$	$-\varepsilon 3^k$	$-\varepsilon 3^k \bar{\omega}$	$-\varepsilon 3^k \omega$	0	$-\frac{q-1}{2}$	$\frac{+1+3^k\sqrt{-3}}{2}$	$\frac{+1-3^k\sqrt{-3}}{2}$
$\overline{\mu_1}$	$\frac{3^k(q-1)}{2}$	$\frac{3^k(q-1)}{2}$	$\frac{-3^k-3^{2k}\sqrt{-3}}{2}$	$\frac{-3^k+3^{2k}\sqrt{-3}}{2}$	$-\varepsilon 3^k$	$-\varepsilon 3^k \omega$	$-\varepsilon 3^k \bar{\omega}$	0	$\frac{q-1}{2}$	$\frac{-1+3^k\sqrt{-3}}{2}$	$\frac{-1-3^k\sqrt{-3}}{2}$
$\overline{\mu_2}$	$\frac{3^k(q-1)}{2}$	$\frac{3^k(q-1)}{2}$	$\frac{-3^k-3^{2k}\sqrt{-3}}{2}$	$\frac{-3^k+3^{2k}\sqrt{-3}}{2}$	$-\varepsilon 3^k$	$-\varepsilon 3^k \omega$	$-\varepsilon 3^k \bar{\omega}$	0	$-\frac{q-1}{2}$	$\frac{+1-3^k\sqrt{-3}}{2}$	$\frac{+1+3^k\sqrt{-3}}{2}$
χ	$3^k(q-1)$	$3^k(q-1)$	$-3^k + 3^{2k}\sqrt{-3}$	$-3^k - 3^{2k}\sqrt{-3}$	$\varepsilon 3^k$	$\varepsilon 3^k \bar{\omega}$	$\varepsilon 3^k \omega$	0	0	0	0
$\bar{\chi}$	$3^k(q-1)$	$3^k(q-1)$	$-3^k - 3^{2k}\sqrt{-3}$	$-3^k + 3^{2k}\sqrt{-3}$	$\varepsilon 3^k$	$\varepsilon 3^k \omega$	$\varepsilon 3^k \bar{\omega}$	0	0	0	0
ψ	$q(q-1)$	$-q$	0	0	0	0	0	0	0	0	0

where $q = 3^{2k+1}$, ε a fixed number from $\{\pm 1\}$, $\omega = e^{2i\pi/3}$ and ξ is a fixed primitive $(q-1)^{th}$ root of unity.

3 The principal 3-block of the Ree groups

Throughout this section, G will denote the small Ree group ${}^2G_2(q)$ of order $q^3(q^3+1)(q-1)$ [15], with $q = 3^{2k+1} \geq 27$. The groups G were first described by Ree [15] and their structure was determined in detail by Ward [17], including most of the character table. Our notation for the conjugacy classes and characters follows the notation introduced in [17]; in particular, m denotes the number 3^k . For the convenience of the reader the character table of ${}^2G_2(q)$, as contained in [17], is given in Table 8, which can be found at the end of this paper.

For k an algebraically closed field of characteristic 3, the group algebra kG decomposes into two blocks: the principal 3-block $B_0(kG)$ and one block of defect zero containing the Steinberg character ξ_3 of degree q^3 (see Table 3).

Table 3: The irreducible characters of G [17]

θ	$\theta(1_G)$	number of characters	θ	$\theta(1_G)$	number of characters
ξ_1	1	1	ξ_9	$m(q^2-1)$	1
ξ_2	q^2-q+1	1	ξ_{10}	$m(q^2-1)$	1
ξ_3	q^3	1	η_r	q^3+1	$(q-3)/4$
ξ_4	$q(q^2-q+1)$	1	η'_r	q^3+1	$(q-3)/4$
ξ_5	$(q-1)m(q+1+3m)/2$	1	η_t	$(q-1)(q^2-q+1)$	$(q-3)/24$
ξ_6	$(q-1)m(q+1-3m)/2$	1	η'_t	$(q-1)(q^2-q+1)$	$(q-3)/8$
ξ_7	$(q-1)m(q+1+3m)/2$	1	η_i^-	$(q^2-1)(q+1+3m)$	$(q-3m)/6$
ξ_8	$(q-1)m(q+1-3m)/2$	1	η_i^+	$(q^2-1)(q+1-3m)$	$(q+3m)/6$

Table 4: Conjugacy classes, their centraliser orders and corresponding defects

conjugacy class	order(g)	$ C_G(g) $	d_g
1_G	1	$ G $	$3(2m+1)$
R^a , $1 \leq a \leq (q-3)/4$	$(q-1)/2$	$q-1$	0
S^a , $1 \leq a \leq (q-3)/24$	$(q+1)/4$	$q+1$	0
V_i , $1 \leq i \leq (q-3m)/6$	$q - \sqrt{3q} + 1$	$q+1-3m$	0
W_i , $1 \leq i \leq (q-3m)/6$	$q + \sqrt{3q} + 1$	$q+1+3m$	0
X	3	q^3	$3(2m+1)$
Y	9	$3q$	$(2m+1)+1$
T	3	$2q^2$	$2(2m+1)$
T^{-1}	3	$2q^2$	$2(2m+1)$
YT	9	$3q$	$(2m+1)+1$
YT^{-1}	9	$3q$	$(2m+1)+1$
JT	6	$2q$	$(2m+1)$
JT^{-1}	6	$2q$	$(2m+1)$
JR^a , $1 \leq a \leq (q-3)/4$	$2 R^a $	$q-1$	0
JS^a , $1 \leq a \leq (q-3)/8$	$2 S^a $	$q+1$	0
J	2	$(q+1)q(q-1)$	$(2m+1)$

Table 4 lists the conjugacy classes of G . Note that for the six families of conjugacy classes, given by R^a , S^a , V_i , W_i , JR^a , JS^a , the elements don't have the order stated, but rather their order divides the given value. Furthermore, the size of the centralisers for these elements are taken from [12, Section 3]. Let

$$\mathcal{S} = \{R^a, S^a, V_i, W_i, JR^a, JS^a\};$$

$|\mathcal{S}| = q - 2$. With slight abuse of notation, we also say that a conjugacy class $\mathcal{C}(g)$ is in \mathcal{S} or a conjugacy class sum $\widehat{\mathcal{C}}(g) \in \mathcal{S}$, if we want to refer to one of these families of conjugacy classes.

As mentioned above, Ward's character table [17] is not quite complete. Some entries are missing; however since only their sum is of interest to us, orthogonality of the columns can be applied to find the required information. The following table provides a list of column orthogonality relations we consider and the implied relation upon character values.

Table 5: Useful orthogonality relations

\mathcal{C}_1 and \mathcal{C}_2	Implication from column orthogonality
R^a, JT	$\sum_r \eta_r(R^a) = \sum_r \eta'_r(R^a)$
$R^a, 1_G$	$\sum_r \eta_r(R^a) = -1$
S^a, Y	$4 - \sum_t \eta_t(S^a) - \sum_t \eta'_t(S^a) = 0$
combined with S^a, J	$\sum_t \eta_t(S^a) = 1, \sum_t \eta'_t(S^a) = 3$
V_i, Y	$\sum_i \eta_i^-(V_i) = 1$
W_i, Y	$\sum_i \eta_i^+(W_i) = 1$
$JR^a, 1$	$\sum_r \eta_r(JR^a) = -\sum_r \eta'_r(JR^a)$
combined with JR^a, JT	$\sum_r \eta_r(JR^a) = -1, \sum_r \eta'_r(JR^a) = 1$
JS^a, Y	$-\sum_t \eta_t(JS^a) - \sum_t \eta'_t(JS^a) = 0$
combined with JS^a, J	$\sum_t \eta_t(JS^a) = 1, \sum_t \eta'_t(JS^a) = -1$

We now write down the two block idempotents, where the equivalence is taken modulo $J(\mathcal{O})G$.

$$\begin{aligned}
\hat{e}_{\xi_3} &= \frac{\xi_3(1)}{|G|} \sum_{g \in G} \xi_3(g^{-1})g \\
&= \frac{q^3}{q^3(q^3+1)(q-1)} \left(q^3 + \sum_{a=1}^{(q-3)/4} \widehat{\mathcal{C}}(R^a) - \sum_{a=1}^{(q-3)/24} \widehat{\mathcal{C}}(S^a) - \sum_{i=1}^{(q-3m)/6} \widehat{\mathcal{C}}(V_i) \right. \\
&\quad \left. - \sum_{i=1}^{(q+3m)/6} \widehat{\mathcal{C}}(W_i) + \sum_{a=1}^{(q-3)/4} \widehat{\mathcal{C}}(JR^a) - \sum_{a=1}^{(q-3)/8} \widehat{\mathcal{C}}(JS^a) + q \cdot \widehat{\mathcal{C}}(J) \right) \\
&\equiv - \sum_{a=1}^{(q-3)/4} \widehat{\mathcal{C}}(R^a) + \sum_{a=1}^{(q-3)/24} \widehat{\mathcal{C}}(S^a) + \sum_{i=1}^{(q-3m)/6} \widehat{\mathcal{C}}(V_i) + \sum_{i=1}^{(q+3m)/6} \widehat{\mathcal{C}}(W_i) \\
&\quad - \sum_{a=1}^{(q-3)/4} \widehat{\mathcal{C}}(JR^a) + \sum_{a=1}^{(q-3)/8} \widehat{\mathcal{C}}(JS^a)
\end{aligned}$$

and as $1 = \hat{e}_0 + \hat{e}_{\xi_3}$, it follows that $\hat{e}_0 = 1 - \hat{e}_{\xi_3} \in \mathcal{O}G$.

3.1 Main Theorem for Ree groups

The aim is to show that the Loewy length of $Z(kGe_0)$ is 3; in particular, over a series of lemmas comprising the rest of this section, the following theorem is proven.

Theorem 3.1. *Let $G = {}^2G_2(q)$ where $q = 3^{2k+1} \geq 27$, and k an algebraically closed field of characteristic 3. Then $LL(Z(kGe_0)) = 3$.*

By Table 4, all non-trivial conjugacy classes have class size divisible by 3 except $\mathcal{C}(X)$ which has size $|\mathcal{C}(X)| = (q^3 + 1)(q - 1)$. Hence a spanning set for $J(Z(kGe_0))$ is given by

$$\mathfrak{D}_G = \{\widehat{\mathcal{C}}(x)e_0 \mid x \in \mathcal{P}, x \neq 1_G, x \notin \mathcal{C}(X)\} \cup \{(\widehat{\mathcal{C}}(X) + 1)e_0\}.$$

Firstly, the multiplication of two conjugacy class sums in \mathfrak{D}_G is considered, ignoring the block idempotent e_0 . For a clear overview, the outcomes of these multiplications are summarised in Table 7.

3.1.1 Computing the product of any two class sums

In this section we compute all the products of two class sums. For most of the algebra structure constants we manipulate the formula of Burnside in such a way so that a minimal amount of computation has to be done; however some of the constants will require a complete calculation of Burnside's formula. Let $cc(G)$ denote the set of conjugacy classes of G .

Before we compute the algebra structure constants, we first need the following lemma which helps us consider the terms arising in $a(x, y, z)$ from the characters $\eta_r, \eta'_r, \eta_t, \eta'_t \eta_l^-$ and η_l^+ as in Table 8 taken from Ward [17].

Lemma 3.2. *Let $\mathcal{C}(y) \in \mathcal{S}$ and $x, z \in {}^2G_2(q)$. Then*

$$\begin{aligned} & \sum_r \eta_r(x) \eta_r(y) \eta_r(z^{-1}) \quad , \quad \sum_r \eta'_r(x) \eta'_r(y) \eta'_r(z^{-1}) \quad , \quad \sum_t \eta_t(x) \eta_t(y) \eta_t(z^{-1}). \\ & \sum_t \eta'_t(x) \eta'_t(y) \eta'_t(z^{-1}) \quad , \quad \sum_l \eta_l^-(x) \eta_l^-(y) \eta_l^-(z^{-1}) \quad \text{and} \quad \sum_l \eta_l^+(x) \eta_l^+(y) \eta_l^+(z^{-1}). \end{aligned}$$

all lie in \mathbb{Z} .

Proof. We shall only consider the case $\mathcal{C}(y) = \mathcal{C}(R^a)$ as the other cases follow by similar arguments. Furthermore, we shall only calculate the following sum for η_r as the situation for η'_r is similar:

$$\sum_r \eta_r(x) \eta_r(R^a) \eta_r(z^{-1}). \quad (3)$$

Note that from the character table, Table 8, the characters η_r takes values inside \mathbb{Z} on elements not lying in $\mathcal{C}(R^a)$ or $\mathcal{C}(JR^a)$. Furthermore from [17, Section I] it follows that $\eta_r(JR^a) = \eta_r(R^a)$ or $-\eta_r(R^a)$ and $\eta_r(R^a) = \epsilon(r^a + r^{-a})$ where $\epsilon = \pm 1$. Using this we obtain the following possibilities for Equation 3.

If x and $z \notin \cup_a (\mathcal{C}(R^a) \cup \mathcal{C}(JR^a))$ then

$$\sum_r \eta_r(x) \eta_r(R^a) \eta_r(z^{-1}) = n \sum_r \eta_r(R^a),$$

for some $n \in \mathbb{Z}$.

If only one of x or z lies in some $\mathcal{C}(R^{a_1}) \cup \mathcal{C}(JR^{a_1})$ then

$$\sum_r \eta_r(x) \eta_r(R^a) \eta_r(z^{-1}) = n \sum_r (\eta(R^{a+a_1}) + \eta_r(R^{a-a_1})),$$

for some $n \in \mathbb{Z}$.

While if $x \in \mathcal{C}(R^{a_1}) \cup \mathcal{C}(JR^{a_1})$ and $z \in \mathcal{C}(R^{a_2}) \cup \mathcal{C}(JR^{a_2})$ then

$$\sum_r \eta_r(x) \eta_r(R^a) \eta_r(z^{-1}) = n \sum_r (\eta(R^{a+a_1+a_2}) + \eta_r(R^{a+a_1-a_2}) + \eta(R^{a-a_1+a_2}) + \eta_r(R^{a-a_1-a_2})),$$

for some $n \in \mathbb{Z}$.

By Table 5 it now follows that Equation 3 evaluates to an element in \mathbb{Z} . □

Lemma 3.3. *Let $\mathcal{C}(x) \in cc(G) \setminus \{X, 1_G\}$ and $\mathcal{C}(y) \in \mathcal{S}$. Then*

$$\widehat{\mathcal{C}}(x) \cdot \widehat{\mathcal{C}}(y) = \begin{cases} e_{\xi_3}, & \text{if } \mathcal{C}(x) \in \mathcal{S} \text{ or } \{\mathcal{C}(J)\}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. First observe that $|C_G(y)|_3 = 1$ and since $q > 3$, $|C_G(x)|_3 \leq q^2$. Thus

$$\left| \frac{q^3(q-1)(q^3+1)}{|C_G(x)||C_G(y)|} \right|_3 \geq q,$$

For the characters ξ_i with $i \neq 3$ or 4, it can be seen that $|\xi_i(1)|_3 < q$ and no value in the corresponding row contains a three in the denominator. Thus any term in $a(x, y, z)$ arising from ξ_i with $i \neq 3$ or 4 is equivalent to zero modulo $J(\mathcal{O})G$.

Next we consider the terms in $a(x, y, z)$ which arise from the characters $\eta_r, \eta'_r, \eta_t, \eta'_t, \eta_l^-, \eta_l^+$. By combining Lemma 3.2 with the additional fact that $|\theta(1)|_3 = 1$ for each such character θ , the terms in $a(x, y, z)$ arising from these characters are equivalent to zero modulo $J(\mathcal{O})G$. In particular, to compute $a(x, y, z)$ modulo $J(\mathcal{O})G$ only $\theta \in \{\xi_3, \xi_4\}$ remain to be considered.

If $\mathcal{C}(x) \notin \{\mathcal{S}, 1_G, \mathcal{C}(J), \mathcal{C}(X)\}$, then $\xi_3(x) = \xi_4(x) = 0$. Hence for $\mathcal{C}(x) \notin \{\mathcal{S}, 1_G, \mathcal{C}(J), \mathcal{C}(X)\}$ it follows that $a(x, y, z) \equiv 0$ modulo $J(\mathcal{O})G$.

Thus it remains to study $\mathcal{C}(x) \in \{\mathcal{S}, \mathcal{C}(J)\}$. In this case $|C_G(x)|_3 \leq q$. Hence as

$$|\xi_4(1)|_3 = q < q^2 \leq \left| \frac{q^3(q-1)(q^3+1)}{|C_G(x)||C_G(y)|} \right|_3,$$

the term for ξ_4 in $a(x, y, z)$ is congruent to zero modulo $J(\mathcal{O})G$. Thus

$$a(x, y, z) \equiv \frac{q^3(q-1)(q^3+1)}{|C_G(x)||C_G(y)|} \left(\frac{\xi_3(x)\xi_3(y)\xi_3(z^{-1})}{q^3} \right).$$

Note the following information about character values of \mathcal{S} . Also note that $\xi_3(J) = q$.

Table 6:

element	$ C_G(g) \bmod 3$	$\xi_3(g)$	coefficient of $\widehat{\mathcal{C}}(g)$ in e_{ξ_3}
R^a	-1	+1	-1
S^a	+1	-1	+1
V_i	+1	-1	+1
W_i	+1	-1	+1
JR^a	-1	+1	-1
JS^a	+1	-1	+1

Hence

$$a(x, y, z) \equiv \begin{cases} q^3 \frac{(-1)^a}{(-1)} \cdot \frac{(-1)^a \xi_3(z^{-1})}{q^3} & \text{if } \mathcal{C}(x) \in \mathcal{S}; \\ q^2 \frac{-1}{(-1)(\pm 1)} \cdot \frac{\mp q \xi_3(z^{-1})}{q^3} & \text{if } \mathcal{C}(x) = \mathcal{C}(J) \end{cases} \equiv -\xi_3(z^{-1}),$$

for $a \in \{1, 2\}$.

Finally, by considering the values ξ_3 takes, we see that

$$a(x, y, z) \equiv -\xi_3(z^{-1}) \equiv \begin{cases} 0 & \text{if } z \notin \mathcal{S}; \\ \text{coefficient of } \widehat{\mathcal{C}}(z^{-1}) \text{ in } e_{\xi_3} & \text{if } z \in \mathcal{S}, \end{cases}$$

where the values for $z \in \mathcal{S}$ follow from the information contained in Table 6. \square

As we have computed $a(x, y, z)$ for $y \in \mathcal{S}$ for all x except $a(X, y, z)$ Proposition 2.4 can now be used to find this final coefficient.

Lemma 3.4. Let $\mathcal{C}(y) \in \mathcal{S}$. Then $\widehat{\mathcal{C}}(X) \cdot \widehat{\mathcal{C}}(y) = e_{\xi_3} - \widehat{\mathcal{C}}(y)$. In particular

$$a(X, y, z) = \begin{cases} 0 & \text{if } y = z \in V_i, W_i, S^a, JS^a; \\ -2 & \text{if } y = z \in R^a, JR^a; \\ \pm 1, & \text{if } y \neq z \text{ and } z \in \mathcal{S}; \\ 0, & \text{if } z \notin \mathcal{S}. \end{cases}$$

Hence $(1 + \widehat{\mathcal{C}}(X)) \cdot \widehat{\mathcal{C}}(y) = e_{\xi_3}$.

Proof. Recall that \mathcal{S} consists of $q-2$ conjugacy classes. For y in \mathcal{S} we have so far calculated all structure constants $a(x, y, z)$ except $a(X, y, z)$. Hence we can use Proposition 2.4 to find these remaining ones. All equivalences are taken modulo $J(\mathcal{O})G$.

We have $|\mathcal{C}(y)| \equiv 0$. Hence by Proposition 2.4, $|\mathcal{C}(y)| = \sum_x a(x, y, z) \equiv 0$. Let $\alpha = a(X, y, z)$. Then

$$\begin{aligned} \sum_x a(x, y, z) &= a(1_G, y, z) + a(J, y, z) + (q-2) \sum_{x \in \mathcal{S}} a(x, y, z) + \alpha \\ &= \begin{cases} 1 + 1 + (q-2)(+1) + \alpha \equiv \alpha, & \text{if } y = z \in V_i, W_i, S^a, JS^a; \\ 1 - 1 + (q-2)(-1) + \alpha \equiv 2 + \alpha, & \text{if } y = z \in R^a, JR^a; \\ 0 + 1 + (q-2)(+1) + \alpha \equiv -1 + \alpha & \text{if } y \neq z \text{ and } z \in V_i, W_i, S^a, JS^a; \\ 0 + (-1) + (q-2)(-1) + \alpha \equiv \alpha + 1 & \text{if } y \neq z \text{ and } z \in R^a, JR^a; \\ \alpha, & \text{if } z \notin \mathcal{S}. \end{cases} \end{aligned}$$

□

We have now dealt with the case that either x or y is taken from \mathcal{S} .

Lemma 3.5. Let $x, y \in \mathcal{C}(Y), \mathcal{C}(YT), \mathcal{C}(YT^{-1}), \mathcal{C}(JT), \mathcal{C}(JT^{-1})$ or $\mathcal{C}(J)$. Furthermore assume that not both x and y are in $\mathcal{C}(J)$. Then $\widehat{\mathcal{C}}(x)\widehat{\mathcal{C}}(y) = 0$.

Proof. We may assume that $\mathcal{C}(y) \neq \mathcal{C}(J)$. By Table 4, it follows that $|C_G(x)|_3$ and $|C_G(y)|_3 \leq 3q$. Hence

$$\left| \frac{q^3(q-1)(q^3+1)}{|C_G(x)||C_G(y)|} \right|_3 \geq \frac{q}{9} = 3^{2k-1}.$$

We assume first that $k \geq 2$. In this case $3^{2k-1} > 3^k$, and thus $|\theta(1)|_3 < \frac{q}{9}$ for $\theta \in \text{Irr}(G)$, unless $\theta = \xi_3$ or ξ_4 . In particular, the corresponding terms in $a(x, y, z)$ for $\theta \neq \xi_3, \xi_4$ are equal to zero modulo $J(\mathcal{O})G$. Furthermore as $\mathcal{C}(y) \in \{\mathcal{C}(Y), \mathcal{C}(YT), \mathcal{C}(YT^{-1}), \mathcal{C}(JT), \mathcal{C}(JT^{-1})\}$, it can be seen that $\xi_3(y) = \xi_4(y) = 0$. Therefore we conclude that $a(x, y, z) \equiv 0$ modulo $J(\mathcal{O})G$.

Now consider the case that $k = 1$. If at most one of x and y lie in $\mathcal{C}(Y), \mathcal{C}(YT), \mathcal{C}(YT^{-1})$, then

$$\left| \frac{q^3(q-1)(q^3+1)}{|C_G(x)||C_G(y)|} \right|_3 \geq \frac{q}{3} = 3^{2k}.$$

Hence the same argument for $k \geq 2$ holds. Thus it remains to assume both x and y lie in $\mathcal{C}(Y), \mathcal{C}(YT)$ and $\mathcal{C}(YT^{-1})$. As $k = 1$, we can explicitly produce the character table of ${}^2G_2(q)$ in GAP [11] and compute for each such x and y that $a(x, y, z) \equiv 0$ modulo $J(\mathcal{O})G$. □

For $y \in \mathcal{C}(Y), \mathcal{C}(YT), \mathcal{C}(YT^{-1}), \mathcal{C}(JT)$ or $\mathcal{C}(JT^{-1})$, the only remaining constants to evaluate are $a(T, y, z), a(T^{-1}, y, z)$ and $a(X, y, z)$. Lemma 2.1 reduces the number of computations required. We state the following values without detail, these were evaluated by computing the full summand in Burnside's formula and are all given modulo $J(\mathcal{O})G$. Note that the computations make use of Table 5, in the cases where $z \in \mathcal{S}$. The full values can be found in [16].

$$\begin{aligned} \widehat{\mathcal{C}}(T) \cdot \widehat{\mathcal{C}}(YT^{-1}) &= 0 &= \widehat{\mathcal{C}}(T^{-1}) \cdot \widehat{\mathcal{C}}(YT) \\ \widehat{\mathcal{C}}(T) \cdot \widehat{\mathcal{C}}(YT) &= 0 &= \widehat{\mathcal{C}}(T^{-1}) \cdot \widehat{\mathcal{C}}(YT^{-1}) \\ \widehat{\mathcal{C}}(T) \cdot \widehat{\mathcal{C}}(JT^{-1}) &= 0 &= \widehat{\mathcal{C}}(T^{-1}) \cdot \widehat{\mathcal{C}}(JT) \\ \widehat{\mathcal{C}}(T) \cdot \widehat{\mathcal{C}}(JT) &= 0 &= \widehat{\mathcal{C}}(T^{-1}) \cdot \widehat{\mathcal{C}}(JT^{-1}) \\ \widehat{\mathcal{C}}(Y) \cdot \widehat{\mathcal{C}}(T) &= 0 &= \widehat{\mathcal{C}}(Y) \cdot \widehat{\mathcal{C}}(T^{-1}) \\ \widehat{\mathcal{C}}(Y) \cdot \widehat{\mathcal{C}}(X) &= 2 \cdot \widehat{\mathcal{C}}(Y) \end{aligned}$$

For $y \in \mathcal{C}(YT), \mathcal{C}(YT^{-1}), \mathcal{C}(JT)$ or $\mathcal{C}(JT^{-1})$ it remains to compute $a(X, y, z)$.

Lemma 3.6. *Let $x \in \mathcal{C}(X)$ and $y \in \mathcal{C}(Y), \mathcal{C}(YT), \mathcal{C}(YT^{-1}), \mathcal{C}(JT)$ or $\mathcal{C}(JT^{-1})$. Then $\widehat{\mathcal{C}}(x) \cdot \widehat{\mathcal{C}}(y) = 2 \cdot \widehat{\mathcal{C}}(y)$.*

Proof. Fix $y \in \mathcal{C}(YT)$; the remaining three cases are proved in the same way.

By Proposition 2.4, $\sum_x a(x, YT, z) = |\mathcal{C}(YT)| = |G|/(3q) \equiv 0 \pmod{3}$. Hence

$$\left(\sum_{x \neq X} a(x, YT, z) \right) + a(X, YT, z) \equiv 0 \pmod{J(\mathcal{O})G} \quad \text{for all } z \in G.$$

Now $a(x \neq X, YT, z) \equiv 0$ except $a(1_G, YT, z)$. However, $a(1_G, YT, z) = 1$ if $z = YT$ and zero otherwise. Hence

$$a(X, YT, z) = \begin{cases} 0, & \text{if } z \notin \mathcal{C}(YT); \\ 2, & \text{if } z \in \mathcal{C}(YT). \end{cases}$$

□

We now consider the case that both x and y lie in $\mathcal{C}(J)$.

Lemma 3.7. *Let x and y both lie in $\mathcal{C}(J)$. Then $\widehat{\mathcal{C}}(x) \cdot \widehat{\mathcal{C}}(y) = \xi_3$.*

Proof. In this case $|C_G(x)|_3 = |C_G(y)|_3 = q$. Thus

$$\left| \frac{q^3(q-1)(q^3+1)}{|C_G(x)||C_G(y)|} \right|_3 = q.$$

Hence for any $\chi \in \text{Irr}(G)$ such that $|\chi(1)|_3 < q$, the corresponding term in $a(x, y, z)$ reduces to zero modulo $J(\mathcal{O})G$. Hence the only terms remaining in $a(x, y, z)$ modulo $J(\mathcal{O})G$ are from ξ_3 and ξ_4 . Using this we see that

$$\begin{aligned} a(J, J, z) &= q \cdot \frac{q^2-q+1}{(q-1)(q+1)} \left(\frac{\xi_3(J)\xi_3(J)\xi_3(z^{-1})}{q^3} + \frac{\xi_4(J)\xi_4(J)\xi_4(z^{-1})}{q(q^2-q+1)} \right) \\ &= q \cdot \frac{q^2-q+1}{(q-1)(q+1)} \left(\frac{\xi_3(z^{-1})}{q} + q \cdot \frac{\xi_4(z^{-1})}{q^2-q+1} \right) \\ &\equiv -\xi_3(z^{-1}) \end{aligned}$$

By Table 6, we have that $\widehat{\mathcal{C}}(J)\widehat{\mathcal{C}}(J) = e_{\xi_3}$.

□

Thus for $y \in \mathcal{C}(J)$ it remains to consider $a(T, y, z), a(T^{-1}, y, z)$ and $a(X, y, z)$. In the following Lemma we deal with T and T^{-1} .

Lemma 3.8. *Let $y \in \mathcal{C}(J)$ and $x \in \mathcal{C}(T)$ or $\mathcal{C}(T^{-1})$. Then $\widehat{\mathcal{C}}(x) \cdot \widehat{\mathcal{C}}(y) = 0$.*

Proof. By applying Theorem 2.3,

$$a(x, y, z) \equiv 0 \pmod{J(\mathcal{O})G} \quad \text{if } z \in \{X, Y, T, T^{-1}, YT, YT^{-1}\};$$

while by [12, Lemma 4.1],

$$a(x, y, z) \equiv 0 \pmod{J(\mathcal{O})G} \quad \text{if } z \in \{R^a, S^a, V_i, W_i\}.$$

Note that the formula used in the calculations by Jones [12], and hence the structure constants calculated, differs up to a scalar; we can adjust appropriately by dividing by $|\mathcal{C}(z)|$.

This leaves $a(x, y, z)$ for $z \in JT, JT^{-1}, JR^a, JS^a$ which can be calculated directly from the character table. In particular

$$\begin{aligned} a(T, J, JT) &= 9 \cdot m^4 + 3 \cdot m^2; \\ a(T, J, JT^{-1}) &= 0; \\ a(T, J, JR^a) &= \frac{9}{2}m^4 - \frac{3}{2}m^2; \\ a(T, J, JS^a) &= \frac{9}{2}m^4 + \frac{3}{2}m^2, \end{aligned}$$

where $m = 3^k$. The equality $a(T^{-1}, J, z) = a(T, J, z^{-1})$ proved in Lemma 2.1 then completes this proof.

□

Now we only have to compute $a(X, J, z)$.

Lemma 3.9. *Let $y \in \mathcal{C}(J)$. Then $\widehat{\mathcal{C}}(X) \cdot \widehat{\mathcal{C}}(J) = e_{\xi_3} - \widehat{\mathcal{C}}(J)$.*

Proof. By Proposition 2.4, $\sum_x a(x, J, z) = |\mathcal{C}(J)| = |G|/((q+1)q(q-1)) \equiv 0 \pmod{3}$. Hence

$$\begin{aligned} \sum_x a(x, J, z) &= a(1_G, J, z) + a(J, J, z) + (q-2)a(x \in \mathcal{S}, J, z) + a(X, J, z) \equiv 0 \\ &= \begin{cases} 1 + 0 + (q-2)(0) + a(X, J, z) \equiv 1 + a(X, J, z), & \text{if } z \in J; \\ 0 + 1 + (q-2)(+1) + a(X, J, z) \equiv -1 + a(X, J, z), & \text{if } z \in V_i, W_i, S^a, JS^a; \\ 0 - 1 + (q-2)(-1) + a(X, J, z) \equiv 1 + a(X, J, z), & \text{if } z \in R^a, JR^a; \\ a(X, J, z), & \text{if } z \notin \mathcal{S}, \mathcal{C}(J). \end{cases} \end{aligned}$$

□

For the remaining structure constants, when $\mathcal{C}(x), \mathcal{C}(y) \in \{\mathcal{C}(T), \mathcal{C}(T^{-1}), \mathcal{C}(X)\}$, direct calculations from the character table are used. As the full Burnside formula was computed, we only provide the final values modulo $J(\mathcal{O})G$. As before, the computations make use of Table 5, in the cases where $z \in \mathcal{S}$. The full values can be found in [16].

$$\begin{aligned} \widehat{\mathcal{C}}(X) \cdot \widehat{\mathcal{C}}(X) &= 2 + \widehat{\mathcal{C}}(X) + \sum \widehat{\mathcal{C}}(R^a) + \sum \widehat{\mathcal{C}}(S^a) + \sum \widehat{\mathcal{C}}(V_i) + \sum \widehat{\mathcal{C}}(W) \\ \widehat{\mathcal{C}}(T) \cdot \widehat{\mathcal{C}}(T) &= 2 \cdot \sum_a \widehat{\mathcal{C}}(R^a) + \sum_a \widehat{\mathcal{C}}(JR^a) + 2 \cdot \sum_a \widehat{\mathcal{C}}(JS^a) \\ &= \widehat{\mathcal{C}}(T^{-1}) \cdot \widehat{\mathcal{C}}(T^{-1}) \\ &= \widehat{\mathcal{C}}(T) \cdot \widehat{\mathcal{C}}(T^{-1}) \\ \widehat{\mathcal{C}}(X) \cdot \widehat{\mathcal{C}}(T) &= 2\widehat{\mathcal{C}}(T) + 2\sum \widehat{\mathcal{C}}(R^a) + \sum \widehat{\mathcal{C}}(JR^a) + 2\sum \widehat{\mathcal{C}}(JS^a) \\ \widehat{\mathcal{C}}(X) \cdot \widehat{\mathcal{C}}(T^{-1}) &= 2\widehat{\mathcal{C}}(T^{-1}) + 2\sum \widehat{\mathcal{C}}(R^a) + \sum \widehat{\mathcal{C}}(JR^a) + 2\sum \widehat{\mathcal{C}}(JS^a) \end{aligned}$$

3.1.2 Summary of multiplications

Table 7: Summary of multiplications of two conjugacy class sums in kG

	R^a	S^a	V_i	W_i	X	Y	T	T^{-1}	YT	YT^{-1}	JT	JT^{-1}	JR^a	JS^a	J
R^a	e_{ξ_3}	e_{ξ_3}	e_{ξ_3}	e_{ξ_3}	γ_1	—	—	—	—	—	—	—	e_{ξ_3}	e_{ξ_3}	e_{ξ_3}
S^a		e_{ξ_3}	e_{ξ_3}	e_{ξ_3}	γ_2	—	—	—	—	—	—	—	e_{ξ_3}	e_{ξ_3}	e_{ξ_3}
V_i			e_{ξ_3}	e_{ξ_3}	γ_3	—	—	—	—	—	—	—	e_{ξ_3}	e_{ξ_3}	e_{ξ_3}
W_i				e_{ξ_3}	γ_4	—	—	—	—	—	—	—	e_{ξ_3}	e_{ξ_3}	e_{ξ_3}
X					α	δ_1	μ	ν	δ_2	δ_3	δ_4	δ_5	γ_5	γ_6	γ_7
Y						—	—	—	—	—	—	—	—	—	—
T							β	β	—	—	—	—	—	—	—
T^{-1}								β	—	—	—	—	—	—	—
YT									—	—	—	—	—	—	—
YT^{-1}										—	—	—	—	—	—
JT											—	—	—	—	—
JT^{-1}												—	—	—	—
JR^a													e_{ξ_3}	e_{ξ_3}	e_{ξ_3}
JS^a														e_{ξ_3}	e_{ξ_3}
J															e_{ξ_3}

In Table 7 we use “–” to denote a zero in kG and

$$\begin{aligned}
\alpha &= 2 + \widehat{\mathcal{C}}(X) + \sum \widehat{\mathcal{C}}(R^a) + \sum \widehat{\mathcal{C}}(S^a) + \sum \widehat{\mathcal{C}}(V_i) + \sum \widehat{\mathcal{C}}(W_i) \\
\beta &= 2 \sum \widehat{\mathcal{C}}(R^a) + \sum \widehat{\mathcal{C}}(JR^a) + 2 \sum \widehat{\mathcal{C}}(JS^a) \\
\gamma_1 &= e_{\xi_3} - \widehat{\mathcal{C}}(R^a) & \delta_1 &= 2 \cdot \widehat{\mathcal{C}}(Y) \\
\gamma_2 &= e_{\xi_3} - \widehat{\mathcal{C}}(S^a) & \delta_2 &= 2 \cdot \widehat{\mathcal{C}}(YT) \\
\gamma_3 &= e_{\xi_3} - \widehat{\mathcal{C}}(V_i) & \delta_3 &= 2 \cdot \widehat{\mathcal{C}}(YT^{-1}) \\
\gamma_4 &= e_{\xi_3} - \widehat{\mathcal{C}}(W_i) & \delta_4 &= 2 \cdot \widehat{\mathcal{C}}(JT) \\
\gamma_5 &= e_{\xi_3} - \widehat{\mathcal{C}}(JR^a) & \delta_5 &= 2 \cdot \widehat{\mathcal{C}}(JT^{-1}) \\
\gamma_6 &= e_{\xi_3} - \widehat{\mathcal{C}}(JS^a) & \mu &= 2\widehat{\mathcal{C}}(T) + 2 \sum \widehat{\mathcal{C}}(R^a) + \sum \widehat{\mathcal{C}}(JR^a) + 2 \sum \widehat{\mathcal{C}}(JS^a) \\
\gamma_7 &= e_{\xi_3} - \widehat{\mathcal{C}}(J) & \nu &= 2\widehat{\mathcal{C}}(T^{-1}) + 2 \sum \widehat{\mathcal{C}}(R^a) + \sum \widehat{\mathcal{C}}(JR^a) + 2 \sum \widehat{\mathcal{C}}(JS^a)
\end{aligned}$$

Note that $\widehat{\mathcal{C}}(X)\widehat{\mathcal{C}}(y) = \gamma_i = e_{\xi_3} - \widehat{\mathcal{C}}(y)$ so that $(1 + \widehat{\mathcal{C}}(X))\widehat{\mathcal{C}}(y) = e_{\xi_3}$. Moreover, $\mathcal{C}(X) \cdot \mathcal{C}(Y) = \delta_i = 2 \cdot \widehat{\mathcal{C}}(y)$ so that $(1 + \widehat{\mathcal{C}}(X))\widehat{\mathcal{C}}(y) = 0$.

Since $e_{\xi_3} \cdot e_0 = 0$, we can see from Table 7 that most pairs of elements in \mathfrak{D}_G multiply to zero. However there exist elements b, b' in \mathfrak{D}_G such that $b \cdot b' \neq 0$:

$$\begin{aligned}
(1 + \widehat{\mathcal{C}}(X))^2 e_0 &= \left(\sum \widehat{\mathcal{C}}(R^a) + \sum \widehat{\mathcal{C}}(S^a) + \sum \widehat{\mathcal{C}}(V_i) + \sum \widehat{\mathcal{C}}(W_i) \right) e_0 \\
&= 2 \sum \widehat{\mathcal{C}}(R^a) + \sum \widehat{\mathcal{C}}(JR^a) + 2 \sum \widehat{\mathcal{C}}(JS^a) \\
(1 + \widehat{\mathcal{C}}(X))\widehat{\mathcal{C}}(T)e_0 &= (1 + \widehat{\mathcal{C}}(X))\widehat{\mathcal{C}}(T^{-1})e_0 = (\widehat{\mathcal{C}}(T^{\pm 1}))^2 e_0 = \widehat{\mathcal{C}}(T) \cdot \widehat{\mathcal{C}}(T^{-1})e_0 \\
&= 2 \sum \widehat{\mathcal{C}}(R^a) + \sum \widehat{\mathcal{C}}(JR^a) + 2 \sum \widehat{\mathcal{C}}(JS^a)
\end{aligned}$$

3.2 The proof of Theorem 3.1

As all the products $\widehat{\mathcal{C}}(x)\widehat{\mathcal{C}}(y)$ have been computed, we can complete the proof of Theorem 3.1.

Proof of Theorem 3.1. By Table 7 and the discussion below it, there exist elements b, b' in \mathfrak{D}_G such that $b \cdot b' \neq 0$. Hence $LL(Z(kGe_0)) \geq 3$.

Note that the number of the conjugacy classes labeled by $\mathcal{C}(S^a)$, for some a , is the only one not congruent to zero modulo 3; in fact, we have $(q-3)/24 \equiv 1$ modulo 3 of those (see Table 4). This explains why $(1 + \widehat{\mathcal{C}}(X))^2 e_0 \neq (1 + \widehat{\mathcal{C}}(X))^2$ while $(1 + \widehat{\mathcal{C}}(X))\widehat{\mathcal{C}}(T^{\pm 1})e_0 = (1 + \widehat{\mathcal{C}}(X))\widehat{\mathcal{C}}(T^{\pm 1})$, and $(\widehat{\mathcal{C}}(T^{\pm 1}))^2 e_0 = (\widehat{\mathcal{C}}(T^{\pm 1}))^2$. In particular,

$$\left(\sum_{a=1}^{(q-3)/24} \widehat{\mathcal{C}}(S^a) \right) \cdot e_{\xi_3} \equiv \left(\sum_{a=1}^{(q-3)/24} \widehat{\mathcal{C}}(S^a) \right) \cdot \left(\sum_{a=1}^{(q-3)/24} \widehat{\mathcal{C}}(S^a) \right) \equiv e_{\xi_3} \neq 0.$$

From the multiplications already computed, it can be concluded that the Loewy length must be equal to 3, since none of the outcomes of the non-zero multiplications of two elements in \mathfrak{D}_G involve the conjugacy classes $\mathcal{C}(X), \mathcal{C}(T), \mathcal{C}(T^{-1})$ or $\mathcal{C}(S^a)$. It therefore follows that any triple of elements in \mathfrak{D}_G will multiply to zero. Hence $LL(Z(kGe_0)) = 3$. \square

4 The 3-block of the Sylow normaliser

We now state the main theorem on the normaliser of a Sylow 3-subgroup. Recall that the group algebra $kN_G(P)$ is indecomposable, where k is an algebraically closed field of characteristic 3. Throughout we shall assume that $q = 3^{2k+1}$ with $k > 0$.

Theorem 4.1. *Let $N = N_G(P)$ where $G = {}^2G_2(q)$, $q = 3^{2k+1} \geq 27$, and $P \in \text{Syl}_3(G)$. Then $LL(Z(kN)) = 2$.*

By Table 1, all non-trivial conjugacy classes of $N_G(P)$ have class size divisible by 3 except $\mathcal{C}(X)$ which has size $|\mathcal{C}(X)| = q - 1$. Therefore a basis for $J(Z(kN_G(P)))$ is given by

$$\mathfrak{B}_{N_G(P)} = \{\widehat{\mathcal{C}}(x) \mid x \in \mathcal{P}, x \neq 1_{N_G(P)}, x \notin \mathcal{C}(X)\} \cup \{\widehat{\mathcal{C}}(X) + 1\}.$$

The proof of Theorem 4.1 will be spread over several lemmas. Let $cc(N_G(P))$ denote the set of conjugacy classes inside $N_G(P)$. All conjugacy class sums are multiplied as elements in $kN_G(P)$, and all equivalences are taken modulo $J(\mathcal{O})N_G(P)$. We will leave the multiplications for the element $\widehat{\mathcal{C}}(X) + 1$ till we have computed the other products.

Firstly we deal with the case where $x \in Ph(w_j)$ not corresponding to J, JT or JT^{-1} .

Lemma 4.2. *Let $\mathcal{C}(x) \in \{Ph(w_j) \mid w_j \neq \pm 1\}$ and $\mathcal{C}(y) \in cc(N_G(P)) \setminus \{\widehat{\mathcal{C}}(1_N), \widehat{\mathcal{C}}(X)\}$. Then $\widehat{\mathcal{C}}(x) \cdot \widehat{\mathcal{C}}(y) = 0$.*

Proof. By Table 1, $|C_G(x)| = q - 1$ and $|C_G(y)|_3 < q^3$. As the only characters which do not vanish on $Ph(w_j)$ are the linear characters α_i for $0 \leq i \leq q - 2$, it follows that for all $z \in N_G(P)$

$$\begin{aligned} a(x, y, z) &= 3^a \cdot \frac{q-1}{s} \left(\sum_{i=0}^{q-2} \alpha_i(x) \alpha_i(y) \alpha_i(z^{-1}) \right) = 3^a \cdot \frac{q-1}{s} \left(\sum_{i=0}^{q-2} \alpha_i(x) \alpha_i(yz^{-1}) \right) \\ &= 3^a \cdot \frac{q-1}{s} (\delta_{x, (zy^{-1})} \cdot |C_{N_G(P)}(x)|), \end{aligned}$$

for $a \geq 1$, $\gcd(3, s) = 1$ and $\delta_{x, y}$ is defined to be equal to 1 if $x = y$ and 0 otherwise. The second equality follows from the fact that degree one characters are representations of the group and the third from the column orthogonality. Thus it follows that $a(x, y, z) \equiv 0$ modulo $J(\mathcal{O})N_G(P)$. \square

As $a(x, y, z) = a(y, x, z)$ it shall be assumed from now on that neither x nor y is of the form $Ph(w_j)$, where $Ph(w_j)$ is not one of J, JT, JT^{-1} . Next consider the conjugacy classes of J, JT and JT^{-1} .

Lemma 4.3. *Let $x \in \mathcal{C}(J), \mathcal{C}(JT)$ or $\mathcal{C}(JT^{-1})$, and $\mathcal{C}(y) \in cc(N_G(P)) \setminus \{\mathcal{C}(1_N), \mathcal{C}(X), \mathcal{C}(Ph(w_j))\}$. Then $\widehat{\mathcal{C}}(x) \cdot \widehat{\mathcal{C}}(y) = 0$.*

Proof. First we observe that

$$\sum_{i=0}^{q-2} \alpha_i(x) \alpha_i(y) \alpha_i(z^{-1}) = \sum_{i=0}^{q-2} (-1)^i \alpha_i(yz^{-1}) = \begin{cases} q-1 & yz^{-1} \in \mathcal{C}(J), \mathcal{C}(JT) \text{ or } \mathcal{C}(JT^{-1}) \\ 0 & \text{otherwise} \end{cases};$$

the final equality follows by taking row orthogonality in the character table of $N_G(P)/P$.

For x as in the statement of the Lemma, we see that

$$\begin{aligned} a(x, y, z) &= \frac{q^3(q-1)}{|C_{N_G(P)}(x)||C_{N_G(P)}(y)|} \left(\sum_{\theta \in \{\mu_i, \overline{\mu_i}\}} \frac{2\theta(x)\theta(y)\theta(z)}{3^k(q-1)} + \sum_{i=0}^{q-2} (-1)^i \alpha_i(yz^{-1}) \right) \\ &= \frac{q^3(q-1)}{3^k(q-1)|C_{N_G(P)}(x)||C_{N_G(P)}(y)|} \left(\sum_{\theta \in \{\mu_i, \overline{\mu_i}\}} 2\theta(x)\theta(y)\theta(z) + 3^k(q-1) \sum_{i=0}^{q-2} (-1)^i \alpha_i(yz^{-1}) \right). \end{aligned}$$

As $\mu_i(g) = \frac{a+b\sqrt{-3}}{2}$ for $a, b \in \mathbb{Z}$ and the summands arising from the α_i add up to an element in \mathbb{Z} , it is enough to consider when the front coefficient, given by

$$\frac{q^3(q-1)}{3^k(q-1)|C_{N_G(P)}(x)||C_{N_G(P)}(y)|},$$

is divisible by 3. However, as $|C_{N_G(P)}(x)| = qs$, where $\gcd(3, s) = 1$, this coefficient reduces to

$$\frac{q^2}{3^k s |C_{N_G(P)}(y)|}.$$

In particular, as $y \notin \mathcal{C}(X)$, it follows that this coefficient is divisible by 3 provided y is not in $\mathcal{C}(T)$ or $\mathcal{C}(T^{-1})$. Thus it remains to consider the cases $y \in \mathcal{C}(T)$ and $\mathcal{C}(T^{-1})$.

For $z \in \mathcal{C}(1_{N_G(P)}), C(X), \mathcal{C}(T), \mathcal{C}(T^{-1}), \mathcal{C}(Y), \mathcal{C}(YT)$ or $\mathcal{C}(YT^{-1})$, we have that $d_x < d_z$ and thus Theorem 2.3 implies $a(x, y, z) \equiv 0$. If $z \in Ph(w_j)$ for $w_j \neq \pm 1$, then

$$a(x, y, z) = 3^a \cdot \frac{q-1}{s} \left(\sum_{i=0}^{q-2} \alpha_i(x) \alpha_i(yz^{-1}) \right),$$

where $a \geq 0$. This sum is non-zero only if $yz^{-1} \in \mathcal{C}(J), \mathcal{C}(JT), \mathcal{C}(JT^{-1})$, which implies that yz^{-1} lies in $Ph(-1)$. As $y \in P$, by taking the image inside $N_G(P)/P = W$, it follows that z must also lie in $Ph(-1)$, which is a contradiction. Thus $a(x, y, z) = 0$.

As $a(x, T, z) = a(x^{-1}, T^{-1}, z^{-1})$ and $\mathcal{C}(J)^{-1} = \mathcal{C}(J)$, $\mathcal{C}(JT)^{-1} = \mathcal{C}(JT^{-1})$, it is enough to consider $y \in \mathcal{C}(T)$ and $z \in \mathcal{C}(J), \mathcal{C}(JT)$ or $\mathcal{C}(JT^{-1})$.

In this case it follows that

$$\sum_{i=0}^{q-2} \alpha_i(x) \alpha_i(T) \alpha_i(z) = \sum_{i=0}^{q-2} \alpha_i(T) = q-1.$$

Therefore, for $a = \frac{-3^k + 3^{2k} \sqrt{-3}}{2}$ and $b = \frac{-1 - 3^k \sqrt{-3}}{2}$,

$$\sum_{\theta \in \{\mu_i, \overline{\mu_i}\}} 2\theta(x)\theta(y)\theta(z) + 3^k(q-1)^2 = \begin{cases} 4(\frac{q-1}{2})^2(a + \overline{a}) + 3^k(q-1)^2 & x, z \in \mathcal{C}(J) \\ 4(\frac{q-1}{2})(a\overline{b} + \overline{a}b) + 3^k(q-1)^2 & x \in \mathcal{C}(J), z \in \mathcal{C}(JT) \\ 4(\frac{q-1}{2})(ab + \overline{a}\overline{b}) + 3^k(q-1)^2 & x \in \mathcal{C}(J), z \in \mathcal{C}(JT^{-1}) \\ 4(b\overline{a} + \overline{b}a) + 3^k(q-1)^2 & x \in \mathcal{C}(JT), z \in \mathcal{C}(JT) \\ 4(bab + \overline{b}\overline{a}\overline{b}) + 3^k(q-1)^2 & x \in \mathcal{C}(JT), z \in \mathcal{C}(JT^{-1}) \\ 4(\overline{b}a\overline{b} + b\overline{a}b) + 3^k(q-1)^2 & x \in \mathcal{C}(JT^{-1}), z \in \mathcal{C}(JT) \end{cases}$$

Hence

$$a(x, T, z) = \begin{cases} 0 & x, z \in \mathcal{C}(J) \\ 0 & x \in \mathcal{C}(J), z \in \mathcal{C}(JT) \\ q & x \in \mathcal{C}(J), z \in \mathcal{C}(JT^{-1}) \\ \frac{q(q-3)}{4} & x \in \mathcal{C}(JT), z \in \mathcal{C}(JT) \\ \frac{q(q-3)}{4} & x \in \mathcal{C}(JT), z \in \mathcal{C}(JT^{-1}) \\ \frac{q(q+1)}{4} & x \in \mathcal{C}(JT^{-1}), z \in \mathcal{C}(JT) \end{cases}$$

By recalling that $a(x, y, z) = a(z^{-1}, y, x^{-1}) \frac{C_H(z)}{C_H(x)}$, we note that $a(JT, T, J) = \frac{2}{q-1} a(J, T, JT^{-1})$, $a(J, T, JT^{-1}) = \frac{2}{q-1} a(J, T, JT)$ and $a(JT, T, JT) = a(JT^{-1}, T, JT^{-1})$. Thus it follows that $a(x, T, z) \equiv 0$ modulo $J(\mathcal{O})N_G(P)$.

This completes the proof. \square

As before, we can now assume that neither x nor y lie in one of $Ph(w_j)$, J, JT or JT^{-1} .

Lemma 4.4. *Let $\mathcal{C}(x), \mathcal{C}(y) \in \{\mathcal{C}(Y), \mathcal{C}(YT), \mathcal{C}(YT^{-1})\}$. Then $\widehat{\mathcal{C}}(x) \cdot \widehat{\mathcal{C}}(y) = 0$.*

Proof. As both $\mathcal{C}(x)$ and $\mathcal{C}(y)$ lie in P , which is a normal subgroup, then $a(x, y, z) = 0$ for $z \in \mathcal{C}(Ph(w_j)), \mathcal{C}(J), \mathcal{C}(JT)$ or $\mathcal{C}(JT^{-1})$. In particular, we may now assume that $\alpha_i(x) = \alpha_i(y) = \alpha_i(z) = 1$.

By Table 1, $|C_G(x)| = |C_G(y)| = 3q$. Hence

$$a(x, y, z) = \frac{q^3(q-1)}{3^2 q^2} \left(\sum_{\theta \in \mathfrak{A}} \frac{\theta(x)\theta(y)\theta(z^{-1})}{\theta(1)} + q-1 + \frac{\lambda(z^{-1})}{q-1} \right).$$

where $\mathfrak{A} = \{\mu_1, \mu_2, \overline{\mu_1}, \overline{\mu_2}, \chi, \overline{\chi}\}$. Note that for $\theta \in \mathfrak{A}$, we have $|\theta(1)|_3 = 3^k$; however at the same time $3^k = |\theta(x)|_3 = |\theta(y)|_3$. Hence $a(x, y, z) \equiv 0$ for all $z \in N_G(P)$. \square

Lemma 4.5. Let $\mathcal{C}(x) \in \{\mathcal{C}(T), \mathcal{C}(T^{-1})\}$ and $\mathcal{C}(y) \in \{\mathcal{C}(Y), \mathcal{C}(YT), \mathcal{C}(YT^{-1})\}$. Then $\widehat{\mathcal{C}}(x) \cdot \widehat{\mathcal{C}}(y) = 0$.

Proof. As $x, y \in P$, which is a normal subgroup, then $a(x, y, z) = 0$ for $z \in \mathcal{C}(Ph(w_j)), \mathcal{C}(J), \mathcal{C}(JT)$ or $\mathcal{C}(JT^{-1})$. If $z \in \mathcal{C}(1_N), \mathcal{C}(X), \mathcal{C}(T)$ or $\mathcal{C}(T^{-1})$, then $d_y < d_z$ and so by Theorem 2.3, $a(x, y, z) \equiv 0 \pmod{J(\mathcal{O})N_G(P)}$.

Thus assume $z \in \mathcal{C}(Y), \mathcal{C}(YT)$ or $\mathcal{C}(YT^{-1})$. Then

$$a(x, y, z) = \frac{(q-1)}{2 \cdot 3} \left((q-1) + 1 + \sum_{\theta \in \mathfrak{A}} \frac{\theta(x)\theta(y)\theta(z^{-1})}{\theta(1)} \right),$$

where $\mathfrak{A} = \{\mu_1, \mu_2, \overline{\mu_1}, \overline{\mu_2}, \chi, \overline{\chi}\}$. The multiplication $\theta(x)\theta(y)\theta(z^{-1})$ is of the form $3^{3k} \cdot s$, where $\gcd(3, s) = 1$. Hence, as $k \geq 1$, we have that modulo $J(\mathcal{O})N_G(P)$

$$a(x, y, z) = \frac{q-1}{2} \left(\frac{q}{3} + 3^{2k-1} \cdot \frac{s_1}{s_2} \right) \equiv 0,$$

as $\gcd(3, s_i) = 1$. □

Lemma 4.6. Let $\mathcal{C}(x), \mathcal{C}(y) \in \{\mathcal{C}(T), \mathcal{C}(T^{-1})\}$. Then $\widehat{\mathcal{C}}(x) \cdot \widehat{\mathcal{C}}(y) = 0$.

Proof. As in the previous lemma, if $z \in \mathcal{C}(Ph(w_j)), \mathcal{C}(J), \mathcal{C}(JT)$ or $\mathcal{C}(JT^{-1})$ then $a(x, y, z) = 0$. If $z \in \mathcal{C}(1_N), \mathcal{C}(X)$ then $d_x < d_z$ and so by Theorem 2.3, $a(x, y, z) \equiv 0$.

If $\mathcal{C}(z) \in \{\mathcal{C}(T), \mathcal{C}(T^{-1}), \mathcal{C}(Y), \mathcal{C}(YT), \mathcal{C}(YT^{-1})\}$ then by Proposition 2.5, $a_{N_G(P)}(x, y, z) \equiv a_G(x, y, z) \equiv 0 \pmod{J(\mathcal{O})N}$. □

Lemma 4.7. Let $\mathcal{C}(y)$ in $cc(N) \setminus \{\mathcal{C}(X), \mathcal{C}(1_N)\}$. Then $(\widehat{\mathcal{C}}(X) + 1) \cdot \widehat{\mathcal{C}}(y) = 0$ and $(\widehat{\mathcal{C}}(X) + 1)^2 = 0$.

Proof. So far we have already calculated all structure constants apart from $a(X, y, z)$; hence we can use Proposition 2.4 to find the remaining ones.

By Proposition 2.4 and the conjugacy class sizes given in Table 1, we have $\sum_x a(x, y, z) = |\mathcal{C}(y)| \equiv 0 \pmod{3}$. Hence modulo $J(\mathcal{O})N_G(P)$.

$$a(1, y, z) + \sum_{x \notin \{1_N, \mathcal{C}(X)\}} a(x, y, z) + a(X, y, z) \equiv 0 \quad \forall z \in N_G(P).$$

Now $a(x \neq X, y, z) \equiv 0$ and $a(1, y, z) \equiv 0$ except $a(1, y, y) = 1$. Therefore

$$a(X, y, z) = \begin{cases} 0, & \text{if } \mathcal{C}(y) \neq \mathcal{C}(z); \\ 2, & \text{if } \mathcal{C}(y) = \mathcal{C}(z). \end{cases}$$

Hence $\widehat{\mathcal{C}}(X) \cdot \widehat{\mathcal{C}}(y) = 2 \cdot \widehat{\mathcal{C}}(y)$ and so $(\widehat{\mathcal{C}}(X) + 1) \cdot \widehat{\mathcal{C}}(y) \equiv 0$.

Moreover $1 + \widehat{\mathcal{C}}(X) = \sum_{\gamma \in Z(P)} \gamma$, and therefore $(1 + \widehat{\mathcal{C}}(X))^2 = q(1 + \widehat{\mathcal{C}}(X)) \equiv 0$. □

This concludes the proof of Theorem 4.1. Moreover, by combining Theorem 4.1 and Theorem 3.1 we have proven the main result of this paper, Theorem 1.1.

Acknowledgments

The work formed part of the second author's PhD research, which was supported by EPSRC grant 1240275. The research of the first author is supported by the LMS Postdoctoral Mobility Grant 15-16 08.

References

- [1] H. I. Blau and G. O. Michler. Modular representation theory of finite groups with T.I. Sylow p -subgroups. *Trans. Amer. Math. Soc.*, 319(2):417–468, 1990.
- [2] M. Broué. Blocs, isométries parfaites, catégories dérivées. *C. R. Acad. Sci. Paris Sér. I Math.*, 307(1):13–18, 1988.
- [3] M. Broué. Isométries parfaites, types de blocs, catégories dérivées. *Astérisque*, (181-182):61–92, 1990.
- [4] W. Burnside. *Theory of groups of finite order*. Dover Publications, Inc., New York, 1955. 2d ed.
- [5] R. Carter. *Simple Groups of Lie Type*. Wiley Classics Library, 1989.
- [6] G. Cliff. On centers of 2-blocks of Suzuki groups. *J. Algebra*, 226(1):74–90, 2000.
- [7] C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.
- [8] C. W. Eaton. Dade’s inductive conjecture for the Ree groups of type G_2 in the defining characteristic. *J. Algebra*, 226(1):614–620, 2000.
- [9] D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 3. Part I. Chapter A*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998. Almost simple K -groups.
- [10] J.B. Gramain. *Generalized Block Theory*. PhD Thesis, 2005.
- [11] The GAP group. *GAP - groups, algorithms and programming, Version 4.7.9; 2015*. (<http://www.gap-system.org>).
- [12] G. A. Jones. Ree groups and Riemann surfaces. *J. Algebra*, 165(1):41–62, 1994.
- [13] G. Karpilovsky. *The Jacobson radical of group algebras*, volume 135 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1987. Notas de Matemática [Mathematical Notes], 115.
- [14] P. Landrock and G. O. Michler. Principal 2-blocks of the simple groups of Ree type. *Trans. Amer. Math. Soc.*, 260(1):83–111, 1980.
- [15] R. Ree. A family of simple groups associated with the simple Lie algebra of type (G_2) . In *Proc. Sympos. Pure Math., Vol. VI*, pages 111–112. American Mathematical Society, Providence, R.I., 1962.
- [16] I. Schwabrow. *The centre of a block*. PhD Thesis, University of Manchester, 2016.
- [17] H. N. Ward. On Ree’s series of simple groups. *Trans. Amer. Math. Soc.*, 121:62–89, 1966.

Appendix

Table 8: Character table of ${}^2G_2(q)$ [17]

	1	$R^a \neq 1$	$S^a \neq 1$	V_i	W_i	X	Y	T	T^{-1}	YT	YT^{-1}	JT	JT^{-1}	$JR^a \neq J$	$JS^a \neq J$	J
ξ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ξ_2	$q^2 - q + 1$	1	3	0	0	$1 - q$	1	1	1	1	1	-1	-1	-1	-1	-1
ξ_3	q^3	1	-1	-1	-1	0	0	0	0	0	0	0	0	1	-1	q
ξ_4	$q(q^2 - q + 1)$	1	-3	0	0	q	0	0	0	0	0	0	0	-1	1	$-q$
ξ_5	$(q-1)m(q+1+3m)/2$	0	1	-1	0	$-(q+m)/2$	m	α	$\bar{\alpha}$	β	$\bar{\beta}$	γ	$\bar{\gamma}$	0	1	$-(q-1)/2$
ξ_6	$(q-1)m(q+1+3m)/2$	0	-1	0	1	$(q+m)/2$	m	α	$\bar{\alpha}$	β	$\bar{\beta}$	$-\gamma$	$-\bar{\gamma}$	0	-1	$(q-1)/2$
ξ_7	$(q-1)m(q+1+3m)/2$	0	1	-1	0	$-(q+m)/2$	m	$\bar{\alpha}$	α	$\bar{\beta}$	β	$\bar{\gamma}$	γ	0	1	$-(q-1)/2$
ξ_8	$(q-1)m(q+1+3m)/2$	0	-1	0	1	$(q+m)/2$	m	$\bar{\alpha}$	α	$\bar{\beta}$	β	$-\bar{\gamma}$	$-\gamma$	0	-1	$(q-1)/2$
ξ_9	$m(q^2 - 1)$	0	0	-1	1	$-m$	$-m$	$\bar{\delta}$	δ	ϵ	$\bar{\epsilon}$	0	0	0	0	0
ξ_{10}	$m(q^2 - 1)$	0	0	-1	1	$-m$	$-m$	$\bar{\delta}$	δ	$\bar{\epsilon}$	ϵ	0	0	0	0	0
η_r	$q^3 + 1$	I-6	0	0	0	1	1	1	1	1	1	1	1	I-6	0	$q+1$
η'_r	$q^3 + 1$	I-6	0	0	0	1	1	1	1	1	1	-1	-1	I-6	0	$-(q+1)$
η_t	$(q-1)(q^2 - q + 1)$	0	II-6	0	0	$2q-1$	-1	-1	-1	-1	-1	-3	-3	0	II-6	$3(q-1)$
η'_t	$(q-1)(q^2 - q + 1)$	0	II-6	0	0	$2q-1$	-1	-1	-1	-1	-1	1	1	0	II-6	$-(q-1)$
η_l^-	$(q^2-1)(q+1+3m)$	0	0	IV-5	0	$-q-1-3m$	-1	$-3m-1$	$-3m-1$	-1	-1	0	0	0	0	0
η_l^+	$(q^2-1)(q+1+3m)$	0	0	0	IV-5	$-q-1+3m$	-1	$3m-1$	$3m-1$	-1	-1	0	0	0	0	0

where $q = 3^{2k+1}$, $m = 3^k$, $\alpha = (-m + im^2\sqrt{3})/2$, $\beta = (-m - im^2\sqrt{3})/2$, $\gamma = (1 - im\sqrt{3})/2$, $\delta = -m + im^2\sqrt{3}$ and $\epsilon = (m + im\sqrt{3})/2$.